Automatica 48 (2012) 297–303

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

On proper refinement of Nash equilibria for bimatrix games ☆

Slim Belhaiza [a,1], Charles Audet [b], Pierre Hansen [c]

[a] King Fahd University of Petroleum and Minerals, Saudi Arabia
[b] GERAD and École Polytechnique de Montréal, Canada
[c] GERAD and École des Hautes Études Commerciales de Montréal, Canada

ARTICLE INFO

Article history:
Received 17 January 2008
Received in revised form
26 March 2011
Accepted 28 July 2011
Available online 22 December 2011

Keywords:
Enumeration
Refinement
Proper
Bimatrix game
Extreme Nash equilibrium
Perfect

Mots clés:
Énumération
Raffinement
Jeu bimatriciel
Équilibre de Nash extrême
Propre

ABSTRACT

In this paper, we introduce the notion of *set of ϵ-proper equilibria* for a bimatrix game. We define a 0–1 mixed quadratic program to generate a sequence of ϵ-proper Nash equilibria and show that the optimization results provide reliable indications on strategy profiles that could be used to generate proper equilibria analytically. This approach can be generalized in order to find at least one proper equilibrium for any bimatrix game. Finally, we define another 0–1 mixed quadratic program to identify non-proper extreme Nash equilibria.

RÉSUMÉ

Dans cet article nous établissons la définition de l'ensemble d'équilibres ϵ-propres pour un jeu bimatriciel. Nous définissons un programme quadratique mixte 0–1 afin de générer une séquence d'équilibres ϵ-propres et de montrer que les résultats de l'optimisation de ce programme permettent d'indiquer les choix stratégiques susceptibles de générer un ou plusieurs équilibres propres analytiquement. Cette approche peut être généralisée afin de trouver au moins un équilibre propre pour tout jeu bimatriciel. Nous définissons aussi un autre programme quadratique mixte 0–1 afin d'identifier les équilibres de Nash non-propres.

1. Introduction

A *bimatrix game* is a strategic confrontation of two players, I and II. A bimatrix game $G(A, B)$ is defined by a pair of $n \times m$ payoff matrices A and B. Each player has a finite number of actions to choose from. The deterministic choice of an action is called *pure strategy*. Player I has to choose between n pure strategies, while player II has to choose between m pure strategies.

Each player attempts to maximize his own payoff by selecting a probability vector over his set of pure strategies. These vectors are combinations of pure strategies, called *mixed strategies*, and represented by probability vectors $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. Hence, player I's payoff is $x_1^t A x_2$ and player II's payoff is $x_1^t B x_2$.

A Nash *equilibrium* is defined as a profile of strategies such that simultaneously, player I maximizes his payoff given the strategic choice of player II and player II maximizes his payoff given the strategic choice of player I. A number of papers have addressed the problem of enumeration of all Nash extreme equilibria for bimatrix games (see Audet, Belhaiza, & Hansen, 2006; Audet, Hansen, Jaumard, & Savard, 2001).

When confronted with a situation where a large number of equilibria can be considered to solve a game, decision makers would have to refine their choices using some other rational concepts in addition to the concept of Nash equilibrium. Perfect and *Proper* equilibria are two refinements of the concept of Nash equilibrium based on the idea that a reasonable equilibrium should be stable against slight perturbations in the equilibrium strategies. It is also well known that a subgame perfect equilibrium for a two-person extensive game corresponds to a proper equilibrium for its corresponding reduced normal form bimatrix game representation. One can find a short review of these concepts at the end of this paper.

Lack of analytical and numerical tools that can be used to generate such equilibria with robustness properties made these refinements rarely used in practice. This paper tries to answer

☆ The material in this paper was partially presented at the 12th Annual Congress of the French National Society of Operations Research and Decision Science (ROADEF 2011), March 2-4, 2011, Saint-Etienne, France. This paper was recommended for publication in revised form under the direction of the Editor, Berç Rüstem.

E-mail addresses: slimb@kfupm.edu.sa, slim.belhaiza@gerad.ca (S. Belhaiza), charles.audet@gerad.ca (C. Audet), pierre.hansen@gerad.ca (P. Hansen).

1 Tel.: +966 38601054; fax: +966 38602340.

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doi:10.1016/j.automatica.2011.07.013

the following question: How can we automatically detect *proper* extreme Nash equilibria?

Section 2 recalls the definition of proper refinement concept and introduces the definition of the *set of ϵ-proper equilibria*. Section 3 proposes a mixed 0–1 quadratic program in order to detect ϵ-proper equilibria. This section details different cases of convergence results and discusses a theoretical procedure to generate proper equilibria and conclude on the non-properness of an equilibrium.

2. Set of ϵ-proper equilibria

The main idea behind the *proper* refinement of Nash equilibria is that a reasonable player would try harder to avoid important mistakes than he or she would try to avoid small ones. While any proper equilibrium profile is perfect, a perfect equilibrium profile could be non-proper. Let us note A_i and A_h respectively as the ith and hth rows of the payoff matrix A. Similarly, we note B_j and B_l respectively as the jth and lth rows of the payoff matrix B.

Definition 2.1. A bimatrix game profile (x_1, x_2) is said to be ϵ-proper equilibrium, for some $\epsilon > 0$, if the following conditions are satisfied:

$$\text{if } A_i x_2 < A_h x_2, \quad \text{then } x_{1i} \le \epsilon x_{1h},\ \forall i, h \in \{1, 2, \ldots, n\}, \tag{2.1}$$

$$\text{if } x_1 B_j < x_1^k B_l, \quad \text{then } x_{2j} \le \epsilon x_{2l},\ \forall j, l \in \{1, 2, \ldots, m\}, \tag{2.2}$$

$$\begin{aligned} &x_{1i} > 0, \quad \forall i \in \{1, 2, \ldots, n\}, \\ &x_{2j} > 0, \quad \forall j \in \{1, 2, \ldots, m\}. \end{aligned} \tag{2.3}$$

To provide a practical tool to identify ϵ-proper equilibria and non-proper equilibria, for any $\epsilon \ge 0$ and $\sigma \ge 0$, we introduce the set

$$\begin{aligned} \Omega_\epsilon^\sigma = \{(x_1, x_2) :\ & \\ &\exists u, v \text{ such that} && \mathbb{1}x_1 = 1,\ \mathbb{1}x_2 = 1, \\ &\sigma \le x_{1i}, && \forall i \in \{1, 2, \ldots, n\}, \\ &\sigma \le x_{2j}, && \forall j \in \{1, 2, \ldots, m\}, \\ &A_h x_2 \le A_i x_2 + L u_{ih}, && \forall i, h \in \{1, 2, \ldots, n\},\ i \ne h, \\ &x_{1i} + u_{ih} \le \epsilon x_{1h} + 1, && \forall i, h \in \{1, 2, \ldots, n\},\ i \ne h, \\ &u_{ih} + u_{hi} \le 1, && \forall i, h \in \{1, 2, \ldots, n\},\ i < h, \\ &u_{ih} \in \{0, 1\}, && \forall i, h \in \{1, 2, \ldots, n\},\ i \ne h, \\ &x_1 B_l \le x_1 B_j + L v_{jl}, && \forall j, l \in \{1, 2, \ldots, m\},\ j \ne l, \\ &x_{2j} + v_{jl} \le \epsilon x_{2l} + 1, && \forall j, l \in \{1, 2, \ldots, m\},\ j \ne l, \\ &v_{jl} + v_{lj} \le 1, && \forall j, l \in \{1, 2, \ldots, m\},\ j < l, \\ &v_{jl} \in \{0, 1\}, && \forall j, l \in \{1, 2, \ldots, m\},\ j \ne l\}. \end{aligned}$$

While u and v are two binary vectors, the parameter $L \in \mathbb{R}^+$ is chosen to be sufficiently large.

The following proposition ensures that each element of Ω_ϵ^σ is an ϵ-proper equilibrium.

Proposition 2.2. *If a strategy profile $(x_1, x_2) \in \Omega_\epsilon^\sigma$ for some $\epsilon > 0$ and $\sigma > 0$, then (x_1, x_2) is an ϵ-proper equilibrium.*

Proof. Suppose that (x_1, x_2) belongs to Ω_ϵ^σ, for some $\epsilon > 0$ and $\sigma > 0$. Let i and h be indices in $\{1, 2, \ldots, n\}$ such that $i \ne h$. Then the inequality $u_{ih} + u_{hi} \le 1$ ensures that the combination $u_{ih} = 1$ and $u_{hi} = 1$ is not possible. Furthermore,

- if $u_{ih} = 0$ and $u_{hi} = 0$ then $A_h x_2 = A_i x_2$.
- if $u_{ih} = 1$, then $\epsilon x_{1i} \le x_{1i} \le \epsilon x_{1h} \le x_{1h}$ implies that $x_{1h} \ge \epsilon x_{1i}$ and $u_{hi} = 0$, thus $A_i x_2 \le A_h x_2$.

It follows that conditions (2.1) are satisfied. In a similar way, conditions (2.2) are satisfied using binary variables v_{jl}, for all $j, l \in \{1, 2, \ldots, m\}$ with $j \ne l$.

Finally, with $0 < \sigma \le x_{2j}$, for all $j \in \{1, 2, \ldots, m\}$, the conditions (2.3) are satisfied. □

Conversely, the following proposition ensures that any ϵ-proper equilibrium belongs to Ω_ϵ^σ for all sufficiently small values of σ.

Proposition 2.3. *If a profile (x_1, x_2) is an ϵ-proper equilibrium for some $\epsilon > 0$, then there exists a $\bar\sigma > 0$ such that $(x_1, x_2) \in \Omega_\epsilon^\sigma$ for every $0 \le \sigma \le \bar\sigma$.*

Proof. If a profile (x_1, x_2) is an ϵ-proper equilibrium for some $\epsilon > 0$, conditions (2.1) can be reformulated using binary variables u_{ih}, for all $i, h \in \{1, 2, \ldots, n\}$, $i \ne h$:

$$\text{If } \begin{cases} A_i x_2 < A_h x_2, \\ x_{1i} \le \epsilon x_{1h}, \end{cases} \text{then } \begin{cases} A_i x_2 \le A_h x_2 + L u_{hi}, \\ x_{1i} + u_{ih} \le \epsilon x_{1h} + 1, \\ u_{hi} = 0, \\ u_{ih} = 1. \end{cases}$$

$$\text{If } \begin{cases} A_h x_2 < A_i x_2, \\ x_{1h} \le \epsilon x_{1i}, \end{cases} \text{then } \begin{cases} A_h x_2 \le A_i x_2 + L u_{ih}, \\ x_{1h} + u_{hi} \le \epsilon x_{1i} + 1, \\ u_{ih} = 0, \\ u_{hi} = 1. \end{cases}$$

$$\text{If } \begin{cases} A_i x_2 = A_h x_2, \\ x_{1i} \le 1, \\ x_{1h} \le 1, \end{cases}$$

$$\text{then } \begin{cases} A_i x_2 \le A_h x_2 + L u_{hi}, & x_{1i} + u_{ih} \le \epsilon x_{1h} + 1, \\ A_h x_2 \le A_i x_2 + L u_{ih}, & x_{1h} + u_{hi} \le \epsilon x_{1i} + 1, \\ u_{hi} = 0, & u_{ih} = 0. \end{cases}$$

In a similar way, conditions (2.2) can be reformulated using binary variables v_{jl}, for all $j, l \in \{1, 2, \ldots, m\}$, $j \ne l$.

And finally, conditions (2.3) ensure that there exists a $\bar\sigma > 0$, such that $\bar\sigma \le x_{1i}$, for all $i \in \{1, 2, \ldots, n\}$ and $\bar\sigma \le x_{2j}$, for all $j \in \{1, 2, \ldots, m\}$.

Then, for every σ such that $0 \le \sigma \le \bar\sigma$ and $\sigma > 0$:

$$\sigma \le x_{1i}, \quad \text{for all } i \in \{1, 2, \ldots, n\},$$
$$\sigma \le x_{2j}, \quad \text{for all } j \in \{1, 2, \ldots, m\}.$$

Thus, $(x_1, x_2) \in \Omega_\epsilon^\sigma$ for every σ, such that $0 \le \sigma \le \bar\sigma$ and $\sigma > 0$. □

Jansen (1993) and Myerson (1978) define a *proper* equilibrium to be the limit of an infinite sequence of ϵ_k-proper equilibria, with ϵ_k converging to zero.

Definition 2.4. An equilibrium $(\hat x_1, \hat x_2)$ is said to be proper if there is a sequence of ϵ_k-proper equilibria (x_1^k, x_2^k) such that

$$\lim_{k\to\infty} \epsilon_k = 0 \quad \text{and} \quad \lim_{k\to\infty} (x_1^k, x_2^k) = (\hat x_1, \hat x_2). \tag{2.4}$$

The main difficulty in applying this definition is to find a convergent sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ of positive real numbers making the sequence $\left\{(x_1^k, x_2^k)\right\}_{k\in\mathbb{N}}$ converge to $(\hat x_1, \hat x_2)$, where (x_1^k, x_2^k) are ϵ_k-proper for each $k \in \mathbb{N}$. However, since Myerson (1978) showed that every bimatrix game possesses at least one proper equilibrium, we can be sure that such a sequence exists for every bimatrix game. In Section 3 we will show how such sequences can be obtained on some examples.

3. Detection of ϵ-proper equilibria

In order to generate such sequence of positive real numbers, we define a family of parametrized mixed 0–1 quadratic programs such that their solutions define a sequence of ϵ-proper equilibria, when the parameter σ converges to 0.

Proposition 3.1. *The perfect equilibrium profile $(\hat x_1, \hat x_2)$ is a proper equilibrium if and only if the following 0–1-mixed quadratic program is feasible for all $\bar\sigma > 0$, and if $\lim_{\sigma\to 0^+} f(\sigma) = 0$.*

$$f(\sigma) = \min_{(x_1,x_2)\in\Omega_\epsilon^\sigma,\epsilon} \epsilon$$
$$\text{s.t.} \quad \hat{x}_{1i} - \epsilon \leq x_{1i} \leq \hat{x}_{1i} + \epsilon, \quad \forall i \in \{1, 2, \ldots, n\}, \quad \hat{x}_{2j} - \epsilon \leq x_{2j} \leq \hat{x}_{2j} + \epsilon, \quad \forall j \in \{1, 2, \ldots, m\}, \quad 0 \leq \epsilon \leq 1. \tag{3.5}$$

Proof. Let $(x_1(\sigma), x_2(\sigma), \epsilon(\sigma))$ be the optimal solution to (3.5) for some given perfect equilibrium profile $(\hat{x}_1, \hat{x}_2)$. Proposition 2.2 ensures that $(x_1(\sigma), x_2(\sigma))$, is an $\epsilon(\sigma)$-proper equilibrium. Conditions (2.4) were reformulated using the minimization of ϵ such that

$$\hat{x}_{1i} - \epsilon \leq x_{1i} \leq \hat{x}_{1i} + \epsilon, \quad \forall i \in \{1, 2, \ldots, n\},$$
$$\hat{x}_{2j} - \epsilon \leq x_{2j} \leq \hat{x}_{2j} + \epsilon, \quad \forall j \in \{1, 2, \ldots, m\},$$

in order to make the ϵ-proper equilibrium converge to $(\hat{x}_1, \hat{x}_2)$.

Hence, if the mixed 0–1 quadratic program (3.5) is feasible for all σ, when $\sigma > 0$ converges to 0, we can conclude from Proposition 2.3 that there is always an ϵ-proper equilibrium $(x_1, x_2) \in \Omega_\epsilon^\sigma$.

Moreover, if the perfect equilibrium $(\hat{x}_1, \hat{x}_2)$ is proper then the optimal objective function value $f(\sigma) = \epsilon(\sigma)$ should necessarily converge to 0, when $\sigma > 0$ converges to 0, to make the solution $(x_1(\sigma), x_2(\sigma))$ converge to $(\hat{x}_1, \hat{x}_2)$ at the same time. One can also notice that $f(0) = 0$.

Else, if $f(\sigma)$ does not converge to 0, when $\sigma > 0$ converges to 0, then such a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper does not exist, when ϵ converges to 0. In this case, it is trivial that the equilibrium point is not proper. □

In conclusion, if $f(\sigma)$ converges to 0, when $\sigma > 0$ converges to 0, it is possible to find a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper converging to $(\hat{x}_1, \hat{x}_2)$, when $\epsilon(\sigma)$ converges to 0.

We use this result by computing the value of $f(\sigma)$ for some small values of σ. The 0–1-mixed quadratic program (3.5) is solved using the NEW-QP algorithm (Perron, 2005). This algorithm is a new version of the QP algorithm (Alarie, Audet, Jaumard, & Savard, 2001). The QP algorithm provides an ξ-optimal solution for feasible quadratic programs, where ξ is the precision parameter. In order to solve the 0–1-mixed quadratic program (3.5) using NEW-QP, we have written the binary value constraints on the u and v variables using the quadratic constraints $u_{ih} - u_{ih}^2 = 0$ and $v_{jl} - v_{jl}^2 = 0$. Because of the discrete values taken by these binary variables, we can be sure that the NEW-QP algorithm provides the optimal solution to the mixed 0–1 quadratic program (3.5). In some cases, the numerical noise which might appear makes it difficult to conclude numerically that an equilibrium is proper. Therefore, it would be risky to use the result provided by the optimization to certify that an equilibrium is proper. However the result of the optimization can be used in order to focalize on some sets of equilibria profiles and analytically find sequences of ϵ-proper equilibria.

Corollary 3.2. *Let $(x_1(\sigma), x_2(\sigma), \epsilon(\sigma))$ be an optimal solution to (3.5) for some $\sigma > 0$. Then $(x_1(\sigma), x_2(\sigma))$ is an $\epsilon(\sigma)$-proper equilibrium, and if $\sigma'' > \sigma' > 0$, then $\epsilon(\sigma'') \geq \epsilon(\sigma') \geq 0$.*

Proof. If $\sigma'' > \sigma' > 0$, the 0–1 mixed quadratic program (3.5) for $\sigma' > 0$ is a relaxation of 0–1 mixed quadratic program (3.5) for $\sigma'' > 0$. In fact the only difference between these two programs is in the constraints of $\Omega_\epsilon^{\sigma'}$ and $\Omega_\epsilon^{\sigma''}$:

$$\sigma' \leq x_{1i}, \quad \forall i \in \{1, 2, \ldots, n\}, \qquad \sigma' \leq x_{2j}, \quad \forall j \in \{1, 2, \ldots, m\}, \quad \text{and}$$
$$\sigma'' \leq x_{1i}, \quad \forall i \in \{1, 2, \ldots, n\}, \qquad \sigma'' \leq x_{2j}, \quad \forall j \in \{1, 2, \ldots, m\}.$$
$$\Rightarrow \begin{array}{ll} \sigma' < \sigma'' \leq x_{1i}, & \forall i \in \{1, 2, \ldots, n\}, \\ \sigma' < \sigma'' \leq x_{2j}, & \forall j \in \{1, 2, \ldots, m\}. \end{array}$$

Thus, $\Omega_\epsilon^{\sigma''} \subseteq \Omega_\epsilon^{\sigma'}$ and $\epsilon(\sigma'') \geq \epsilon(\sigma') \geq 0$. □

There are two possible outcomes when evaluating $f(\sigma)$ for some small values of σ. The first possibility is that $f(\sigma)$ appears to converge to zero. The second possibility is that $f(\sigma)$ appears to be bounded below by some strictly positive value, say $\bar{\epsilon}$.

3.1. *Case 1: $f(\sigma)$ converges to zero*

This numerical result is not enough to conclude on the properness of the equilibrium profile. However, we can use it as an indication to find proper equilibria by focusing on some profiles. In fact, if $f(\sigma)$ converges to zero one can conclude that there exists at least one sequence of ϵ-proper equilibria that is very close to the equilibrium profile being tested for properness. We can then analytically find a sequence of ϵ-proper equilibria that converges to the equilibrium profile. As shown by Myerson (1997) this can be performed by iteratively satisfying ϵ-proper equilibrium conditions (Definition 2.1). The following example shows how this procedure can be applied.

Example 3.3. The following (5×5) bimatrix game has 7 extreme Nash equilibria identified in Table 1.

$$A = B = \left(\begin{array}{c|ccccc} & y_1 & y_2 & y_3 & y_4 & y_5 \\ \hline x_1 & 2 & 4 & 5 & 6 & 7 \\ x_2 & 2 & 1 & 0 & 8 & 1 \\ x_3 & 2 & 5 & 6 & 0 & 1 \\ x_4 & 0 & 2 & 5 & 4 & 7 \\ x_5 & 2 & 3 & 6 & 5 & 7 \end{array}\right).$$

We have used the algorithms $E\chi MIP$ (Audet et al., 2006) to enumerate all seven extreme Nash equilibria of this game.

This game has four maximal Nash subsets $T_1 = \{1, 2, 6\}$, $T_2 = \{3, 4\}$, $T_3 = \{5\}$ and $T_4 = \{7\}$.

The optimization results in Table 2 indicate that there exist sequences of ϵ-proper equilibria close to extreme equilibria 3, 5, 6 and 7. We will use the information provided by these extreme equilibria to analytically generate such sequences.

a. *Equilibrium* 3

With extreme equilibrium 3 strategy profile player 1 plays only x_3 and player 2 plays only y_3. For player 1, a sequence of ϵ-proper equilibria would then take into account that x_3 is his best choice and the probability of playing x_3 should be very close to 1. At the same time for player 2, a sequence of ϵ-proper equilibria would then take into account that y_3 is his best choice and the probability of playing y_3 should be very close to 1. According to the payoff matrix, player 1 has to choose between:

$$Ay = \begin{pmatrix} 2y_1+ & 4y_2+ & 5y_3+ & 6y_4+ & 7y_5 \\ 2y_1+ & y_2+ & & 8y_4+ & y_5 \\ 2y_1+ & 5y_2+ & 6y_3+ & & y_5 \\ & 2y_2+ & 5y_3+ & 4y_4+ & 7y_5 \\ 2y_1+ & 3y_2+ & 6y_3+ & 5y_4+ & 7y_5 \end{pmatrix}.$$

Since player 1 would have to consider x_3 as his first best, x_5 should be his second best (because y_3 is very close to 1). Thus

$$2y_1 + 5y_2 + 6y_3 + y_5 > 2y_1 + 3y_2 + 6y_3 + 5y_4 + 7y_5$$
$$\Rightarrow 2y_2 > 5y_4 + 7y_5.$$

Therefore, $y_2 > y_4$ and $y_2 > y_5$. Player 2 should have incentive to give more probability to y_2 compared to y_4 and y_5. According to the payoff matrix, player 2 has to choose between:

$$xB = \begin{pmatrix} 2x_1+ & 2x_2+ & 2x_3+ & & 2x_5 \\ 4x_1+ & x_2+ & 5x_3+ & 2x_4+ & 3x_5 \\ 5x_1+ & & 6x_3+ & 5x_4+ & 6x_5 \\ 6x_1+ & 8x_2+ & & 4x_4+ & 5x_5 \\ 7x_1+ & x_2+ & x_3+ & 7x_4+ & 7x_5 \end{pmatrix}^t.$$

Table 1
Extreme Nash equilibria for (5×5) bimatrix game.

Eq.	x					y					α	β
1	0	0	0	0	1	0	0	0	0	1	7	7
2	0	0	0	1	0	0	0	0	0	1	7	7
3	0	0	1	0	0	0	0	1	0	0	6	6
4	0	0	1/6	0	5/6	0	0	1	0	0	6	6
5	0	1	0	0	0	0	0	0	1	0	8	8
6	1	0	0	0	0	0	0	0	0	1	7	7
7	7/8	1/8	0	0	0	0	0	0	3/4	1/4	25/4	25/4

Table 2
Example (5×5).

Eq.	Perfect	Proper	$\bar{\epsilon}$	$\bar{\sigma}$	Quasi-strong	Isolated	Regular
1	Yes	No	0.2894	5×10^{-3}	No	No	No
2	No	No	0.7325	10^{-3}	No	No	No
3	Yes	Yes	0.05627	10^{-5}	No	No	No
4	Yes	No	0.2000	10^{-6}	Yes	No	No
5	Yes	Yes	0.0564	10^{-5}	Yes	Yes	Yes
6	Yes	Yes	0.054	10^{-5}	No	No	No
7	Yes	Yes	0.0776	6×10^{-5}	Yes	Yes	Yes

Since x_3 is very close to 1, one can observe that player 2 has indeed good incentive to prefer y_2 to y_4 and y_5 because the payoff provided by these strategies are:

$$4x_1 + x_2 + 5x_3 + 2x_4 + 3x_5 > 6x_1 + 8x_2 + 4x_4 + 5x_5$$

and

$$4x_1 + x_2 + 5x_3 + 2x_4 + 3x_5 > 7x_1 + x_2 + x_3 + 7x_4 + 7x_5.$$

In order to comply with the conditions of Definition 2.1, player 2 can play for example: $(y_1 = \frac{1}{2}\epsilon^2,\ y_2 = \frac{1}{2}\epsilon,\ y_3 = 1 - \frac{1}{2}(\epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4),\ y_4 = \frac{1}{2}\epsilon^4,\ y_5 = \frac{1}{2}\epsilon^3)$.

At the same time, player 1 can play for example:

$$\left(x_1 = \frac{1}{2}\epsilon^2,\ x_2 = \frac{1}{2}\epsilon^4,\ x_3 = 1 - \frac{1}{2}(\epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4),\right.$$
$$\left. x_4 = \frac{1}{2}\epsilon^3,\ x_5 = \frac{1}{2}\epsilon\right).$$

It is now made clear that we have a sequence of ϵ-proper equilibria that converges to the extreme equilibrium 3 when ϵ converges to 0.

b. *Equilibrium* 6

The same procedure applied to the extreme equilibrium 6 suggests that one possible sequence of ϵ-proper equilibria that converges to the extreme equilibrium 6 is:

$$\left(x_1 = 1 - \frac{2}{3}(\epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4),\ x_2 = \frac{2}{3}\epsilon^3,\ x_3 = \frac{2}{3}\epsilon^4,\right.$$
$$\left. x_4 = \frac{2}{3}\epsilon^2,\ x_5 = \frac{2}{3}\epsilon\right)$$

and

$$\left(y_1 = \frac{2}{3}\epsilon^4,\ y_2 = \frac{2}{3}\epsilon^3,\ y_3 = 1 - \frac{2}{3}\epsilon^2,\ y_4 = \frac{2}{3}\epsilon,\right.$$
$$\left. y_5 = 1 - \frac{2}{3}(\epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4)\right).$$

These analytical results are confirmed also by the optimization results presented in Table 2.

Regarding extreme equilibria 5 and 7, we can similarly find such sequences of ϵ-proper equilibria. Moreover these two extreme Nash equilibria are found to be *regular*. Regular equilibria have all kind of robustness properties including properness. Jansen (1987) showed that an equilibrium point of a bimatrix game is regular if and only if it is isolated and quasi-strong. One can find at the end of this paper a short review of these refinements of the Nash equilibrium concept.

c. *Equilibrium* 5

$C(x_1) = \{2\}$ and $C(x_2) = \{4\}, M(A, x_2) = \{2\}$ and $M(x_1, B) = \{4\} \Rightarrow$ quasi-strong. The determinant of $\big(8\big)$is equal to $8 \neq 0 \Rightarrow$ isolated. This equilibrium is regular, essential, perfect and proper.

d. *Equilibrium* 7

$C(x_1) = \{1, 2\}$ and $C(x_2) = \{4, 5\}, M(A, x_2) = \{1, 2\}$ and $M(x_1, B) = \{4, 5\} \Rightarrow$ quasi-strong. The determinant of $\begin{pmatrix} 6 & 7 \\ 8 & 1 \end{pmatrix}$ is equal to $-50 \neq 0 \Rightarrow$ isolated. This equilibrium is also regular, essential, perfect and proper.

As in Audet, Belhaiza, and Hansen (2010) we have used a pair of linear programs to conclude on the perfectness of each extreme equilibrium.

We conclude this example by providing two other sequences of ϵ-proper equilibria converging to non-extreme equilibria of this game.

Using the extreme equilibria 3 and 4, if player 1 has to randomize on strategies x_3 and x_5, in order to comply with the conditions of Definition 2.1 player 2 would have to play such that $2y_2 = 5y_4 + 7y_5$. It means that player 2 would be indifferent between y_2 and y_4 or between y_2 and y_5.

The first case is only possible when $4x_1+x_2+5x_3+2x_4+3x_5 = 6x_1 + 8x_2 + 4x_4 + 5x_5$ which yields $x_3 = \frac{2}{5}x_1 + \frac{7}{5}x_2 + \frac{2}{5}x_4 + \frac{2}{5}x_5$. Since $x_3 + x_5$ is expected to be very close to 1 one can conclude that x_5 should be very close to $\frac{5}{7}$ while x_3 should be very close to $\frac{2}{7}$. Thus player 2 would have to order his best strategies in the following order $G(y_3) > G(y_5) > G(y_4) > G(y_2) > G(y_1)$. This strategic order is impossible because $2y_2$ would be less than $5y_4 + 7y_5$.

The second case is only possible when $4x_1 + x_2 + 5x_3 + 2x_4 + 3x_5 = 7x_1 + x_2 + x_3 + 7x_4 + 7x_5$ which yields $x_3 = \frac{3}{4}x_1 + \frac{5}{4}x_4 + x_5$. Since $x_3 + x_5$ is expected to be very close to 1 one can conclude that x_3 and x_5 should be very close to $\frac{1}{2}$. Thus, player 2 would have to order his best strategies in the following order $G(y_3) > G(y_5) = G(y_2) > G(y_4) > G(y_1)$. This strategic order is possible when player 2 plays for example:

$$\left(y_1 = \frac{1}{4}\epsilon^3,\ y_2 = \frac{7}{8}\epsilon + \frac{5}{8}\epsilon^2,\right.$$
$$\left. y_3 = 1 - \frac{9}{8}\epsilon - \frac{7}{8}\epsilon^2 - \frac{1}{4}\epsilon^3,\ y_4 = \frac{1}{4}\epsilon^2,\ y_5 = \frac{1}{4}\epsilon\right)$$

and player 1 plays for example:

$$\left(x_1 = \frac{3}{8}\epsilon^2,\ x_2 = \frac{3}{8}\epsilon^3,\ x_3 = \frac{1}{2} - \frac{3}{64}\epsilon + \frac{3}{64}\epsilon^2 - \frac{3}{16}\epsilon^3,\right.$$
$$\left. x_4 = \frac{3}{8}\epsilon,\ x_5 = \frac{1}{2} - \frac{21}{64}\epsilon - \frac{27}{64}\epsilon^2 - \frac{3}{16}\epsilon^3\right)$$

which converges to the proper equilibrium

$$\left(x_1 = 0,\ x_2 = 0,\ x_3 = \frac{1}{2},\ x_4 = 0,\ x_5 = \frac{1}{2}\right) \quad \text{and}$$
$$(y_1 = 0,\ y_2 = 0,\ y_3 = 1,\ y_4 = 0,\ y_5 = 0).$$

Using the extreme equilibria 1 and 6 by randomizing on strategies x_1 and x_5 for player 1, we also find the proper equilibrium

$$\left(x_1 = \frac{1}{2},\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,\ x_5 = \frac{1}{2}\right) \quad \text{and}$$
$$(y_1 = 1,\ y_2 = 0,\ y_3 = 0,\ y_4 = 0,\ y_5 = 0).$$

e. *Discussion*

Following the analysis of this game one can ask: "Do all bimatrix games have at least one extreme proper equilibrium"? The answer is "No". One can find many bimatrix games in the literature where all proper equilibria found are not extreme.

In such a case we can easily prove that there exists at least one pair of perfect extreme equilibria belonging to the same Selten subset that could be used to find a sequence of ϵ-proper equilibria converging to a proper equilibrium. This is mainly due to the fact that if no extreme proper equilibrium can be found, we still know there exists at least one proper equilibrium for the bimatrix game and this proper equilibrium is by definition also perfect. This proper and perfect equilibrium can then be obtained by a convex combination of at least a pair of extreme perfect equilibria belonging to the same Selten subset. Since Borm, Jansen, Potters, and Tijs (1993) proved that any extreme perfect equilibrium is also an extreme equilibrium, if no extreme proper equilibrium is found we can always find at least a pair of perfect extreme equilibria that could be used to generate a proper equilibrium. The following example illustrates this case.

Example 3.4. In this zero-sum bimatrix game we have two extreme Nash equilibria:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$\{X = (0, 1)$ with $Y = (1, 0, 0)\}$ and $\{X = (1, 0)$ with $Y = (1, 0, 0)\}$. These two extreme equilibria are perfect but none of them is proper. In fact the optimization of the corresponding quadratic programs (3.5) shows that $\bar{\epsilon} = \min f(\sigma)$ converges to $\frac{1}{2}$ when σ converges to zero. By randomizing over the two strategies of player 1 we find the following sequence of ϵ-proper equilibria:

$$X = \left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{and} \quad Y = \left(1-\epsilon, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right).$$

The generalization of this procedure makes it possible to define an algorithmic approach to find a proper equilibrium for any bimatrix game:

Step 1. Enumerate all extreme Nash equilibria.

Step 2. Identify all Nash maximal subsets.

Step 3. Identify extreme perfect equilibria and maximal Selten subsets.

Step 4. For each extreme perfect equilibrium generate the convergence results of the corresponding quadratic program (3.5).

Table 3
Extreme Nash equilibria for Myerson (1997).

Eq.	x_1				x_2		α_1	α_2
1	0	0	1	0	1	0	6	6
2	0	1	0	0	0	1	4	4
3	0	1	0	0	1/3	2/3	4	4
4	1	0	0	0	0	1	4	4
5	1	0	0	0	1/3	2/3	4	4

Step 5. If an extreme equilibrium appears very close to a sequence of ϵ-proper equilibria find such a sequence analytically.

Step 6. Else randomize on the strategy profiles of extreme perfect equilibria (belonging to the same Selten subset) closest to a sequence of ϵ-proper equilibria to find such a sequence analytically.

3.2. *Case 2: $f(\sigma) \geq \bar{\epsilon}$*

The case where $f(\sigma)$ appears to be bounded below by some strictly positive value $\bar{\epsilon}$ implies that there are no ϵ-proper equilibrium near $(\hat{x}_1, \hat{x}_2)$ for values of ϵ less than $\bar{\epsilon}$, and therefore $(\hat{x}_1, \hat{x}_2)$ would not be proper.

In (3.5), let us suppose that $f(\sigma)$ converges to $\bar{\epsilon} > 0$, when $\sigma > 0$ converges to 0. We define a 0–1 mixed quadratic program with the same conditions as Ω, with $\epsilon \leq \bar{\epsilon}/2$ and maximizing σ. If the optimal objective function of this program is equal to *zero* we can conclude that it would be impossible to find a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper converging to this equilibrium. Therefore the equilibrium is not proper.

Theorem 3.5. *If the optimal objective value of the following* 0–1 *mixed quadratic program*

$$\begin{array}{lll} \max\limits_{(x_1,x_2)\in\Omega^{\sigma}_{\epsilon},\epsilon,\sigma} & \sigma & \\ \text{s.t.} & \hat{x}_{1i} - \epsilon \leq x_{1i} \leq \hat{x}_{1i} + \epsilon, & \\ & \quad \forall i \in \{1, 2, \ldots, n\}, & \\ & \hat{x}_{2j} - \epsilon \leq x_{2j} \leq \hat{x}_{2j} + \epsilon, & (3.6) \\ & \quad \forall j \in \{1, 2, \ldots, m\}, & \\ & 0 \leq \epsilon \leq \bar{\epsilon}/2 & \end{array}$$

is zero for some $\bar{\epsilon} > 0$, then the equilibrium $(\hat{x}_1, \hat{x}_2)$ is not proper.

Proof. If the optimal objective value is equal to 0, it is impossible to find a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper converging to $(\hat{x}_1, \hat{x}_2)$. The equilibrium $(\hat{x}_1, \hat{x}_2)$ is not proper. □

With this result, automatic detection of non-proper extreme Nash equilibria can be carried out over any set of extreme Nash equilibria of a bimatrix game.

The first example shows how the objective function does not converge to zero in the case of a non-perfect equilibrium.

Example 3.6. Let A and B be the payoff matrices of a bimatrix game taken from Myerson (1997)

$$A = \begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 6 & 3 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 6 & 0 \\ 0 & 2 \end{pmatrix}.$$

Both algorithms $E\chi MIP$ Audet et al. (2006); Audet, Belhaiza, and Hansen (2009) and *EEE* Audet et al. (2001) enumerated five extreme Nash equilibria (Table 3).

As mentioned by Myerson (1997), the first extreme Nash equilibrium is the only proper equilibrium of this game. While the optimal values of ϵ seem to converge to $\bar{\epsilon} = 0.618$, as σ approaches 0 (Fig. 1) with non-perfect extreme equilibria 2, 3, 4 and 5. We define a 0–1 mixed quadratic program with the same

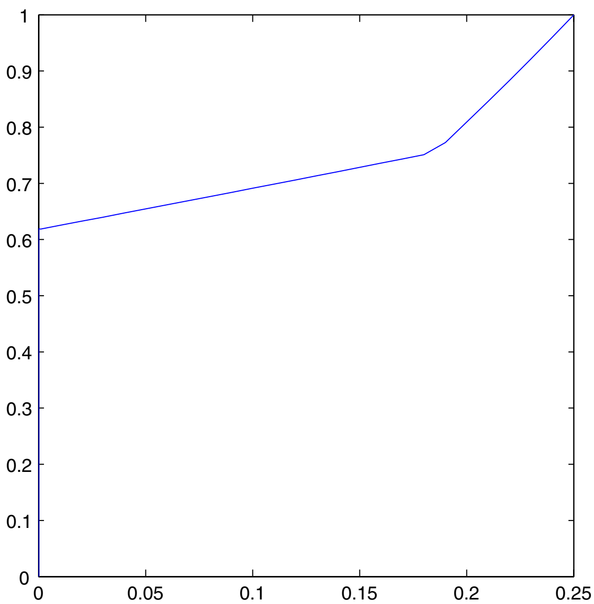

Fig. 1. Plot of $\epsilon = \min f(\sigma)$.

conditions as in (3.6), with $\epsilon \le \bar{\epsilon}/2$ and maximizing σ. Such a 0–1 mixed quadratic program has an optimal objective equal to zero.

The set of extreme proper Nash equilibria defines the set of extreme points of all *Maximal Myerson sets* (Jansen, 1993). There is only one maximal Myerson subset for the bimatrix game taken from Myerson (1997).

4. Conclusion

In this paper we presented a mathematical programming approach for the refinement of Nash equilibria. After complete enumeration of all extreme Nash equilibria, ϵ-proper sequences of equilibria are found using the indications provided by the convergence numerical results of a 0–1 mixed quadratic program. Even in the worst case where no extreme proper equilibrium is found, we have shown that we can always find a pair of extreme perfect equilibria belonging to the same Selten subset in order to find a proper equilibrium. Finally, non-proper extreme Nash equilibria are found using the result of another 0–1 mixed quadratic program. One can conclude that these results could be useful to generate subgame perfect equilibria for two-person extensive games.

Appendix

A.1. Extreme Nash equilibrium

The set *NE* of all equilibrium points of a bimatrix game is the union of a finite number of polytopes called *maximal Nash subsets* (Millham, 1974). A subset $T \subset NE$ is a Nash subset if and only if every pair of elements in T is interchangeable:

$$(x_1, x_2) \in T,\ (y_1, y_2) \in T \Leftrightarrow (x_1, y_2) \in T \quad \text{and} \quad (y_1, x_2) \in T.$$

A Nash subset T is called maximal if it is not properly contained in another Nash subset (Jansen, 1993). Each extreme point of one of these maximal Nash subsets is called *extreme Nash equilibrium*. Each Nash equilibrium can be obtained by a convex combination of some extreme Nash equilibria.

A.2. Perfect equilibrium

According to Myerson (1997) and Selten (1975) there is always at least one perfect equilibrium for any strategic form game.

Definition A.1. Let $(\hat{x}_1, \hat{x}_2)$ be a Nash equilibrium of a bimatrix game $G(A, B)$. If there is a unit vector x_1 such that $x_1 A \ge \hat{x}_1 A$ and $x_1 A \ne \hat{x}_1 A$, or if there is a unit vector x_2 such that $Bx_2 \ge B\hat{x}_2$ and $Bx_2 \ne B\hat{x}_2$ then $(\hat{x}_1, \hat{x}_2)$ is not perfect. Otherwise, $(\hat{x}_1, \hat{x}_2)$ is said to be perfect.

In other words, every perfect equilibrium is undominated.

A.3. Essential equilibrium

According to Wu and Jiang (1972) the *Essential* refinement is based on the concept of stability of an equilibrium against slight perturbations in the payoffs of the game.

Definition A.2. A strategy profile (x_1, x_2) is an essential equilibrium of a bimatrix game $G(A, B)$ if there exists, with every neighborhood N_x of (x_1, x_2) a neighborhood N_G of (A, B) such that $G(A', B')$ has no equilibria in N_x for all $(A', B') \in N_G$.

It is known that every essential equilibrium is perfect (van Damme, 1983). Jansen (1981) paid special attention to equilibrium points that are *Quasi-strong* and *isolated* at the same time; these equilibria were found to be essential.

A.4. Quasi-strong equilibrium

For an equilibrium profile (x_1, x_2) of a bimatrix game $G(A, B)$, let $N = \{1, \dots, n\}$ and $M = \{1, \dots, m\}$. Then $M(A, x_2)$ is defined as the set of pure best replies of player I against x_2:

$$M(A, x_2) = \{i \in N;\ e_i A x_2 = \max_{k \in N} e_k A x_2\}, \tag{A.7}$$

and similarly,

$$M(x_1, B) = \{j \in M;\ x_1 B e_j = \max_{k \in M} x_1 B e_j\}, \tag{A.8}$$

is the set of pure best replies of player II against x_1 (Harsanyi, 1973).

The *carrier* of x_1, $C(x_1)$ is the set $\{i \in N;\ x_{1i} > 0\}$ and carrier of x_2, $C(x_2)$ is the set $\{j \in M;\ x_{2j} > 0\}$.

Definition A.3. Any equilibrium profile (x_1, x_2) is *quasi-strong* if

$$C(x_1) = M(A, x_2) \quad \text{and} \quad C(x_2) = M(x_1, B).$$

Jansen (1981) showed that a quasi-strong and *isolated* equilibrium point is stable against slight perturbations of the payoffs of the game.

A.5. Isolated equilibrium

An equilibrium profile (x_1, x_2) of a bimatrix game $G(A, B)$ is said to be *isolated* if there exists a neighborhood N_x of (x_1, x_2) such that it is the only equilibrium of $G(A, B)$ in this neighborhood N_x. In other words, any isolated equilibrium is an extreme equilibrium defining an isolated maximal Nash subset. Enumeration of all maximal Nash subsets can be used in order to automatically detect isolated equilibria. Moreover, Jansen (1981) proposed the following definition.

Definition A.4. Let (x_1, x_2) be a quasi-strong equilibrium of a bimatrix game $G(A, B)$ with $A, B > 0$. Then (x_1, x_2) is isolated if and only if $|C(x_1)| = |C(x_2)|$ and the matrices $[a_{ij}]_{i \in C(x_1), j \in C(x_2)}$ and $[b_{ij}]_{i \in C(x_1), j \in C(x_2)}$ are nonsingular.

While this definition applies only for bimatrix games $G(A, B)$ such that $A, B > 0$, it is well known that every bimatrix game can be modified in order to make $A, B > 0$ and without changing the set of maximal Nash subsets. For example, this can easily be done by adding $1 + |a_{\min}|$, with $a_{\min} = \min\ a_{ij}$, to each element of A and $1 + |b_{\min}|$,with $b_{\min} = \min\ b_{ij}$, to each element of B.

Jansen (1981) points out that an isolated equilibrium is essential if and only if it is quasi-strong. Moreover, van Damme (1983) showed that an isolated and quasi-strong equilibrium point is perfect and proper. This was also obtained by Okada (1984) for bimatrix games.

A.6. Regular equilibrium

For any *Regular* (Jansen, 1981, 1987) equilibrium we can conclude that it is proper. A *Regular* equilibrium profile was first defined by Harsanyi (1973) such that the Jacobian of a mapping associated with the game evaluated at the equilibrium point is non-singular. This definition was later improved by van Damme (1983) for a two-person case. He proved that an equilibrium is regular if and only if it is *quasi-strong* and *isolated* and showed that such equilibria are *strongly stable* and proper.

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Slim Belhaiza is an Assistant Professor of Mathematics at the King Fahd University of Petroleum and Minerals. His research interests include the development of algorithms for Game theory and Vehicle Routing. He obtained a Ph.D. degree in applied mathematics from the École Polytechnique de Montréal in 2008, and worked for an optimization company in Montréal from 2008 to 2009.

Charles Audet is a Professor of Mathematics at the École Polytechnique de Montréal.

His research interests include the analysis and development of algorithms for structured global optimization, and blackbox nonsmooth optimization. He obtained a Ph.D. degree in applied mathematics from the École Polytechnique de Montréal in 1998, and worked as a post-doc at the Rice University in Houston, Texas from 1998 to 2000.

Pierre Hansen obtained a Ph.D. degree in Mathematics, from the University of Brussels in 1974. He has taught in Belgium, France, USA, Canada, and for short periods in Italy, Germany, Hong Kong, China and Brazil.

Hansen is currently a Professor and holder of the Data Mining Chair at the HEC Montréal. He is the recipient of several research prizes including the EURO Gold Medal, 1986, the Merit Award of the Canadian Operational Research Society, 1999, and the Pierre Rousseau Prize of ACFAS 2008. He is an author, and most of the time co-author with colleagues and students, of more than 300 papers in refereed journals from various fields. Hansen is a Fellow of the Royal Society of Canada, 1999. He is also a member of the International Academy of Mathematical Chemistry, 2005.