CHAPTER TWO
LIMITS AND CONTINUITY

2.1 LIMITS *(AN INTUITIVE APPROACH)*

Objectives:

- *One sided limits*
- *The Relationship Between One-Sided limit and Two-Sided Limits*
- *Infinity limit and Vertical Asymptotes*
- *Limit at Infinity and Horizontal Asymptotes*

2.1.1 LIMITS (AN INFORM VIEW). *If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write*

$$\lim_{x \to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a is L".

2.1.2 ONE- SIDED LIMITS

$$\lim_{x \to a^+} f(x) = L$$

which is read "*the limit of* $f(x)$ *as x approaches a from the right is L*"

$$\lim_{x \to a^-} f(x) = L$$

which is read "*the limit of* $f(x)$ *as x approaches a from the left is L*".

2.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS

$\lim_{x \to a} f(x) = L$ *if and only if* $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

2.1.5 DEFINITION. *A line* $x = a$ *is called a vertical asymptote of the graph of a function f if* $f(x) \to +\infty$ *or* $f(x) \to -\infty$ *as x approaches a from the left or right.*

2.1.7 DEFINITION. *A line* $y = L$ *is called a horizontal asymptote of the graph of a function f if* $\lim_{x \to +\infty} f(x) = L$ *or* $\lim_{x \to -\infty} f(x) = L$

2.2 COMPUTING LIMITS

Objectives:

- *Some Basic Limits*
- *Limits of Polynomials and Rational Functions as* $x \to a$
- *Indeterminate Forms of Type 0/0*
- *Limits involving Radicals*
- *Limits of Piecewise-Defined Functions*

2.2.1THEOREM. *Let a and k be real numbers.*

$$\lim_{x \to a} k = k \qquad \lim_{x \to a} x = a$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = +\infty$$

2.2.2 THEOREM. *Let a be a real number, and suppose that* $\lim_{x\to a} f(x) = L_1$ *and* $\lim_{x\to a} g(x) = L_2$. *Then:*

$$a)\ \lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$$

$$b)\ \lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$$

$$c)\ \lim_{x\to a}[f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1 L_2$$

$$d)\ \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2},\ \textit{provided}\ L_2 \neq 0$$

$$e)\ \lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$$

REMARK:

$$a)\ \lim_{x\to a}\left[f(x)+g(x)+h(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) + \lim_{x\to a} h(x)$$

$$b)\ \lim_{x\to a}\left[f(x)g(x)h(x)\right] = \lim_{x\to a} f(x)\lim_{x\to a} g(x)\lim_{x\to a} h(x)$$

$$c)\ \lim_{x\to a} x^n = \left(\lim_{x\to a} x\right)^n = a^n$$

$$d)\ \lim_{x\to a}\left(k\cdot f(x)\right) = k\cdot\lim_{x\to a} f(x)$$

2.2.3 THEOREM. *For any polynomial*

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number a,

$$\lim_{x \to a} p(x) = c_0 + c_1 a + \cdots c_n a^n = p(a)$$

2.2.4 THEOREM. *Consider the rational function*

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ *and* $q(x)$ *are polynomials. For any real number a,*

$a)$ *if* $q(a) \neq 0$, *then* $\lim_{x \to a} f(x) = f(a) = \dfrac{p(a)}{q(a)}$

$b)$ *if* $q(a) = 0$ *but* $p(a) \neq 0$, *then* $\lim_{x \to a} f(x)$ *does not exist*

2.3 COMPUTING LIMITS: END BEHAVIOR

Objectives:

- *Some Basic Limits*
- *Limits* x^n *as* $x \to \pm\infty$
- *Limits of Polynomials as* $x \to \pm\infty$
- *Limits of Rational Functions as* $x \to \pm\infty$
- *Limits Involving Radicals*

2.3.1 THEOREM. *Let k be a real number*

$$\lim_{x\to-\infty} k = k \qquad \lim_{x\to\infty} k = k$$

$$\lim_{x\to-\infty} x = -\infty \qquad \lim_{x\to\infty} x = \infty$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0 \qquad \lim_{x\to\infty} \frac{1}{x} = 0$$

2.3.2 THEOREM. *Suppose that*

$$\lim_{x\to\infty} f(x) = L_1 \quad \textit{and} \quad \lim_{x\to\infty} g(x) = L_2. \textit{ Then:}$$

$$a)\ \lim_{x\to\infty}[f(x)+g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x) = L_1 + L_2$$

$$b)\ \lim_{x\to\infty}[f(x)-g(x)] = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x) = L_1 - L_2$$

$$c)\ \lim_{x\to\infty}[f(x)g(x)] = \left(\lim_{x\to\infty} f(x)\right)\left(\lim_{x\to\infty} g(x)\right) = L_1 L_2$$

$$d)\ \lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} = \frac{L_1}{L_2}, \textit{ provided } L_2 \neq 0$$

$$e)\ \lim_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\infty} f(x)} = \sqrt[n]{L_1}$$

REMARK:

$$a)\ \lim_{x\to\infty} \left(f(x)\right)^n = \left(\lim_{x\to\infty} f(x)\right)^n$$

$$b)\ \lim_{x\to\infty} \frac{1}{x^n} = \left(\lim_{x\to\infty} \frac{1}{x}\right)^n = 0 \qquad \lim_{x\to-\infty} \frac{1}{x^n} = \left(\lim_{x\to-\infty} \frac{1}{x}\right)^n = 0$$

$$c)\ \lim_{x\to\infty} \left(k \cdot f(x)\right) = k \cdot \lim_{x\to\infty} f(x)$$

REMARK:

$$a)\ \lim_{x\to+\infty} x^n = +\infty, \quad n = 1,2,3,\ldots$$

$$b)\ \lim_{x\to-\infty} x^n = \begin{cases} -\infty, & n = 1,3,5,\ldots \\ +\infty, & n = 2,4,6,\ldots \end{cases}$$

$$If\ \ c_n \neq 0\ \ then$$

$$c)\ \lim_{x\to-\infty} (c_0 + c_1 x + \cdots + c_n x^n) = \lim_{x\to-\infty} c_n x^n$$

$$d)\ \lim_{x\to+\infty} (c_0 + c_1 x + \cdots + c_n x^n) = \lim_{x\to+\infty} c_n x^n$$

2.4 LIMITS (DISCUSSED MORE RIGOROUSLY)

Objectives:

- *Motivation For The Definition of Limit.*

2.4.1 LIMIT DEFINITION. Let $f(x)$ be defined for all x in some open interval containing the number a, with the possible exception that $f(x)$ need not be defined at a. We will write

$$\lim_{x\to a} f(x) = L$$

if given any number $\varepsilon > 0$ *we can find a number* $\delta > 0$ *such that* $|f(x) - L| < \varepsilon \;\; \textit{if} \;\; |x - a| < \delta$

2.5 CONTINUITY

Objectives:

- *Definition of Continuity*
- *Continuity on an Interval and Continuity of Polynomials*
- *Some Properties of Continuous Functions*
- *Continuity of Rational Functions*
- *Continuity of Compositions*
- *Continuity From The Left and Right*
- *The Intermediate-Value Theorem*

2.5.1 DEFINITION. *A function f is said to be continuous at* $x = c$ *provided the following conditions are satisfied:*

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

2.5.2 THEOREM. *Polynomials are Continuous everywhere.*

2.5.3 THEOREM. *If the functions f and g are continuous at c, then*

a) $f + g$ *is continuous at c.*

b) $f - g$ *is continuous at c.*

c) $f\,g$ *is continuous at c.*

d) *f/g is continuous at c if* $g(c) \neq 0$.

2.5.4 THEOREM. *A rational function is continuous at every number where the denominator is nonzero.*

2.5.5 THEOREM. *If* $\lim_{x \to c} g(x) = L$ *and if the function f is continuous at L, then* $\lim_{x \to c} f(g(x)) = f(L)$. *That is*

$$\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x)).$$

2.5.6 THEOREM.

a) *If the function g is continuous at c, and the function f is continuous at g(c), then the composition fog is continuous at c.*

b) *If the function g is continuous everywhere, and the function f is continuous everywhere, then the composition fog is continuous everywhere.*

2.5.7 DEFINITION. *A function f is said to be continuous on a closed interval* $[a,b]$ *if the following conditions are satisfied:*

1 *f is continuous on* (a,b).

2 *f is continuous from the right at a.*

3 *f is continuous from the left at b.*

2.5.8 THEOREM. (*Intermediate-Value Theorem*). *If f is continuous on closed interval* $[a,b]$ *and k is any number between* $f(a)$ *and* $f(b)$, *then there is at least one number x in the interval* $[a,b]$ *such that* $f(x)=k$.

2.5.9 THEOREM. *If f is continuous on* $[a,b]$, *and if* $f(a)$ *and* $f(b)$ *are nonzero have opposite signs, then there is at least one solution of the equation* $f(x)=0$ *in the interval* (a,b).

2.6 LIMITS AND CUNTINUITY OF TRIGONOMETRIC FUNCTION

Objectives:

- *Continuity of Trigonometric Functions*
- *Obtaining Limits by Squeezing*

2.6.1 THEOREM. *If c is any number in the interval domain of the stated trigonometric function, then*

$$\lim_{x\to c} \sin x = \sin c \qquad \lim_{x\to c} \cos x = \cos c \qquad \lim_{x\to c} \tan x = \tan c$$

$$\lim_{x\to c} \csc x = \csc c \qquad \lim_{x\to c} \sec x = \sec c \qquad \lim_{x\to c} \cot x = \cot c$$

2.6.2 THEOREM (*The Squeezing Theorem*). *Let f,g, and h be functions satisfying*

$$g(x) \le f(x) \le h(x)$$

for all x in some interval containing the number c, with the possible exception that the inequalities need not hold at c. If g and h have the same limit as x approaches c, say

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then f has also this limit as x approaches c, that is,

$$\lim_{x\to c} f(x) = L$$

2.6.3 THEOREM.

$$a)\ \lim_{x\to 0}\frac{\sin x}{x}=1 \qquad b)\ \lim_{x\to 0}\frac{1-\cos x}{x}=0$$