Section 3.11 *Hyperbolic functions*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. know the hyperbolic functions
2. understand fundamental hyperbolic identities
3. differentiate hyperbolic & inverse hyperbolic functions

Definition of hyperbolic functions

- Hyperbolic functions are defined using e^x and e^{-x}

The six hyperbolic functions are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

See examples 1, 2 done in class

Fundamental hyperbolic identities

- Like trigonometric identities, the hyperbolic identities are very useful in simplifying expressions and questions involving differentiation, integration etc.
- The fundamental hyperbolic identities are as follows.

$$\cosh x + \sinh x = e^{x}, \qquad \cosh x - \sinh x = e^{-x}$$

$$\cosh(-x) = \cosh x, \qquad \sinh(-x) = -\sinh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \text{sech}^2 x$$

$$\sinh 2x = 2\sinh x \cosh x, \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh 2x = 2\sinh^2 x + 1$$
$$\cosh 2x = 2\cosh^2 x + 1$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$
$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

All of the above and similar identities can easily be proved by using the definitions of hyperbolic functions.

See examples 3, 4 done in class

Derivatives of hyperbolic functions

- The differentiation formulas for hyperbolic functions can easily be derived by differentiating the R.H.S. of definition of hyperbolic functions.

Differentiation formulas for hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

See examples 5, 6 done in class

Derivatives of inverse hyperbolic functions

- Derived from the differentiation formulas of hyperbolic functions and using implicit differentiation. See class explanation for more.

Differentiation formulas for inverse hyperbolic functions

$$\frac{d}{dx}\left(\sinh^{-1} u\right)=\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\cosh^{-1} u\right)=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\tanh^{-1} u\right)=\frac{1}{1-u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\coth^{-1} u\right)=\frac{1}{1-u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1} u\right)=\frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1} u\right)=\frac{-1}{u\sqrt{1+u^2}}\frac{du}{dx}$$

Note:
The last two formulas have “minus” sign

See example 7, 8 done in class

End of 3.11