Section 2.2 *The limit of a function*

Learning outcomes

After completing this section, you will inshaAllah be able to

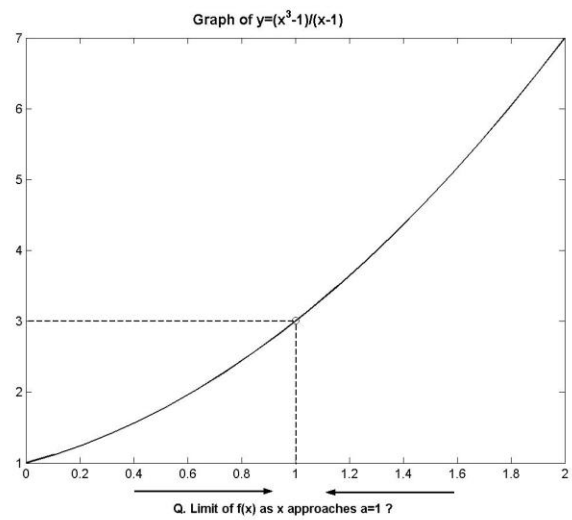
1. an idea about the meaning and definition of limit
2. get an idea about the meaning of one-sided limit
3. know the meaning of existence of limit
4. understand and compute infinite limits
5. find vertical asymptotes of a function

Meaning of limit

- We learn by an example

Example: To understand $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

- We use the graph of the function and a table of values near x=1
- By graph



Note: The function $y = \frac{x^3 - 1}{x - 1}$ is not defined at x=1. In fact the circle 'o' in the graph indicates that this point is missing from the graph.

Question

What happens to the values of

$y = \frac{x^3 - 1}{x - 1}$ as x gets closer to 1

- By table

A table of values of the function near x=1 is

x	0.5	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25	1.5
y	1.750	2.313	2.710	2.970	2.997	not defined	3.003	3.030	3.310	3.813	4.750

- Conclusion: As the values of x get closer and closer to 1, we see that the values of y get closer and closer to 3.
- That means $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3.$

See class explanation for more understanding

(Specially "Run-and-Hit idea")

Understanding the definition of limit

This is read as

"limit of $f(x)$ is L, as x approaches a"

We say that

$$\lim_{x \to a} f(x) = L$$

if

we can make $f(x)$ as close to L as we like

by taking x sufficiently close to a

For practical purpose, it is best to use

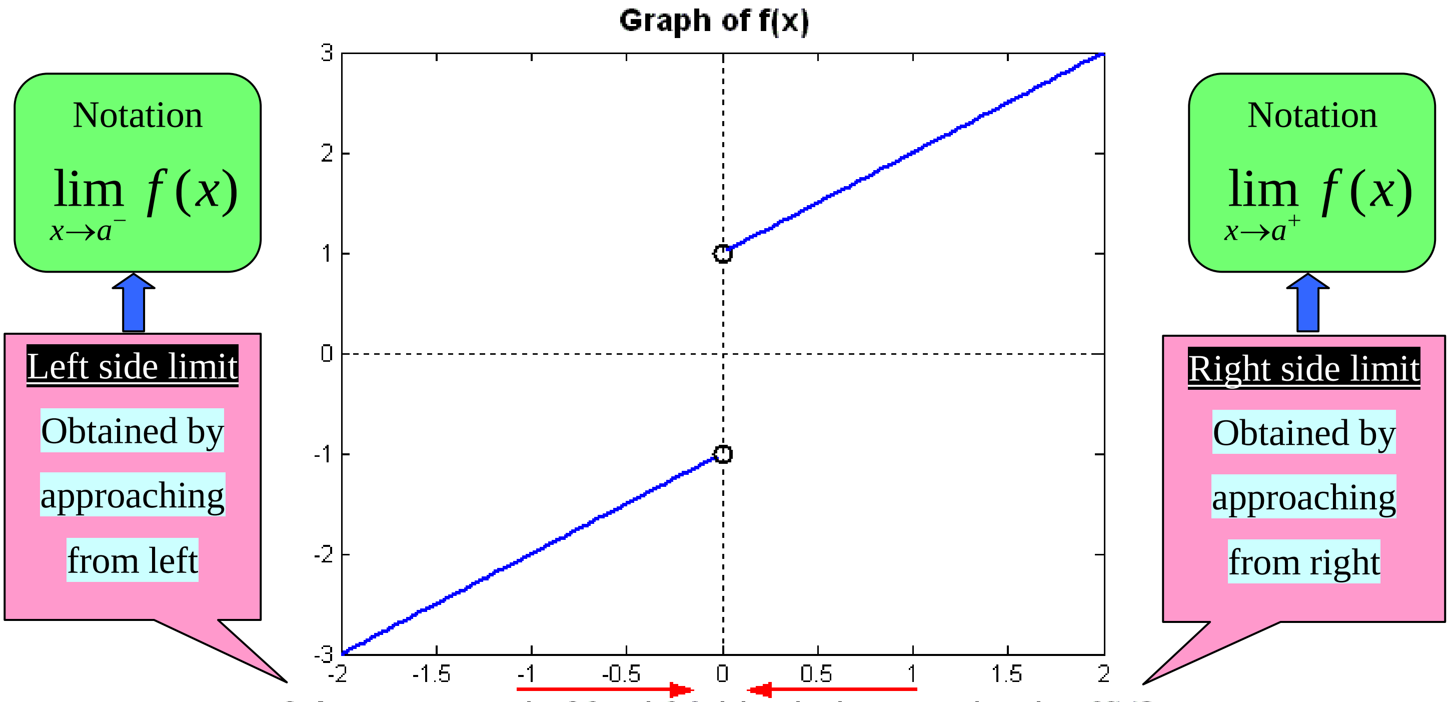
Run – and – Hit Idea

explained in class

See example 1 done in class

One-sided limits

- We learn by an example
- Consider a function $y = f(x)$ given by the following graph.



Q. As we move toward a=0 from left & right, what happens to the value of f(x)?

- From the graph we have:
  - When we approach 0 from the left side, the value of *f(x)* approaches –1.
  - When we approach 0 from the right side, the value of *f(x)* approaches 1.
- We describe this situation by saying
  - The limit of *f(x)* is –1 as x approaches 0 from left and write as

$$\lim_{x\to 0^-} f(x) = -1$$, and

  - The limit of *f(x)* is 1 as x approaches 0 from right and write as

$$\lim_{x\to 0^+} f(x) = 1$$

Q. Will we always get different answers of left and right limits?

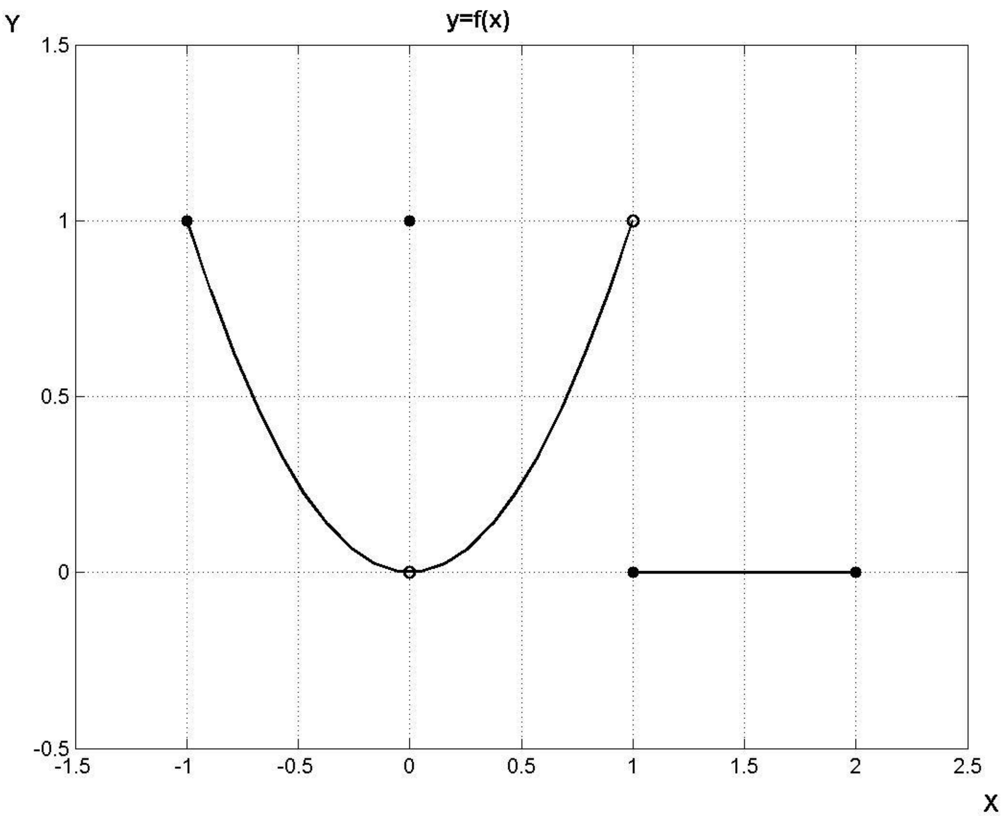
Ans. No. Check what happened in example 1 above.

Meaning of existence of a limit

- We see from above that some time the left side & right side limit will be same, and some time these will be different.

> - $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = L$ if
>
>   left side limit = right side limit
>
>   i.e. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$

Example:



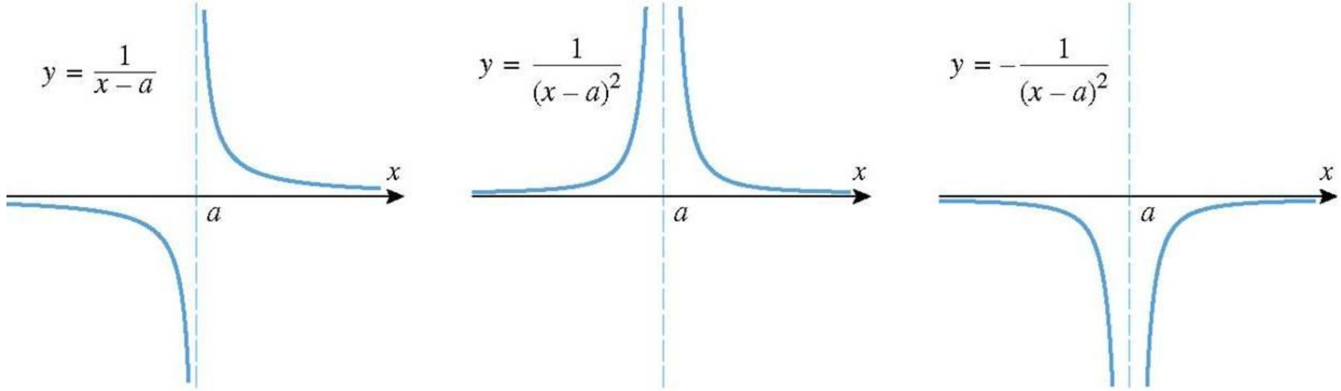
Q: Does $\lim_{x\to 0} f(x)$ exist?

Q. Does $\lim_{x\to 1} f(x)$ exist?

See examples 2, 3 done in class

What are infinite limits?
(those limits whose answer is ∞ or $-\infty$)

- Look at the following graphs to understand the meaning of infinite limits



$$\lim_{x \to a^+} \frac{1}{x-a} = +\infty$$
$$\lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

$$\lim_{x \to a} \frac{1}{(x-a)^2} = +\infty$$

$$\lim_{x \to a} -\frac{1}{(x-a)^2} = -\infty$$

See class discussion & explanation

Computing infinite limits $\lim_{x \to a} f(x)$
i.e. the answer is ∞ or $-\infty$

This generally happens when directly substituting $x = a$ gives $\left(\frac{k}{0}\right)$ form $(k \neq 0)$

Trick

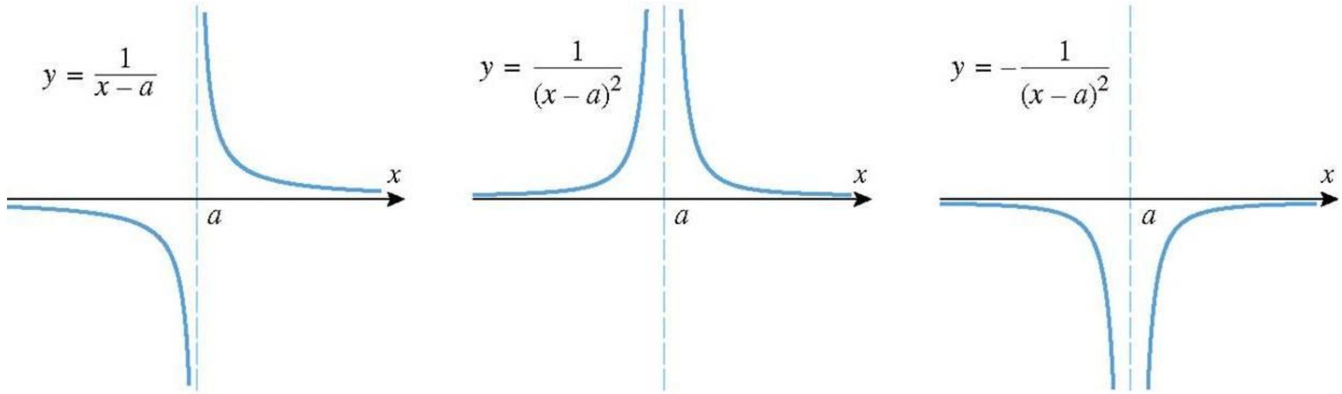
Look at the sign and get the answer as ∞ or $-\infty$

Note usefulness of factorization

See examples 4, 5, 6, 7 done in class

Vertical Asymptotes

- Look at the following graphs again.



$\lim_{x \to a^+} \frac{1}{x-a} = +\infty$

$\lim_{x \to a^-} \frac{1}{x-a} = -\infty$

$\lim_{x \to a} \frac{1}{(x-a)^2} = +\infty$

$\lim_{x \to a} -\frac{1}{(x-a)^2} = -\infty$

What's special about line x=a

It runs (very close &) parallel to graph up to $\pm\infty$

What happens to graph when we get near the value x=a

The graph either shoots up to ∞ or shoots down to $-\infty$

A vertical line $x = a$ is called **vertical asymptote** of graph of $f(x)$ if one of the following is true

- $\lim_{x \to a^-} f(x) = \infty$ or $-\infty$
- $\lim_{x \to a^+} f(x) = \infty$ or $-\infty$
- $\lim_{x \to a} f(x) = \infty$ or $-\infty$

Special situation for rational functions

Vertical asymptotes for $f(x) = \frac{P(x)}{Q(x)}$

- $x = a$ is a vertical asymptote if $Q(a) = 0$ and $P(a) \neq 0$

But it is better to compute limits to get complete idea of the situation

Note usefulness of factorization

See examples 8, 9 done in class

End of 2.2