

1. Use the graph of $f(x)$ below to answer the following:

(a) (5 points) Evaluate the limit if it exists. If it does not exist, explain why. Use the symbols ∞ or $-\infty$ as appropriate.

1 pt. i. $\lim_{x \rightarrow 3^+} f(x) = \infty$

1 pt. ii. $\lim_{x \rightarrow \infty} f(x) = 1$

1 pt. iii. $\lim_{x \rightarrow -3} f(x) = -4$

1 pt. iv. $\lim_{x \rightarrow 0} f(x) = \text{D.N.E.}$ ($\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$)

1 pt. v. $\lim_{x \rightarrow 5} f(x) = 0$

(b) (4 points) At which values of x , if any, is f discontinuous? Why?

1 pt. ① $x = 0$ (Jump)

1 pt. ② $x = 3$ ($f(3)$ is undefined)

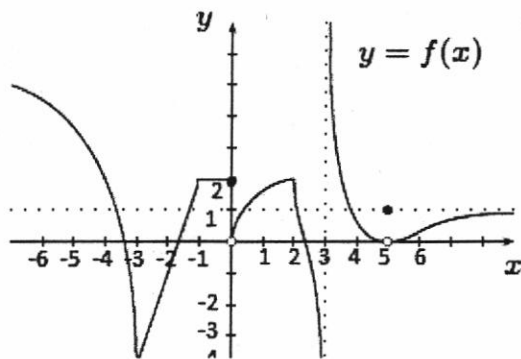
2 pt. ③ $x = 5$ ($\lim_{x \rightarrow 5} f(x) \neq f(5)$)

(c) (1 point) State the equations of any horizontal asymptotes.

$$y = 1$$

(d) (1 point) State the equations of any vertical asymptotes.

$$x = 3$$



2. Evaluate the limit if it exists. If it does not exist, explain why. Use the symbols ∞ or $-\infty$ as appropriate.

$$(a) \text{ (2 points) } \lim_{x \rightarrow e} (\pi^{x-e} + 2 \ln x) = \underbrace{\pi^{e-e}}_{1 \text{ pt}} + 2 \ln e = \boxed{3} \quad 1 \text{ pt.}$$

$$(b) \text{ (4 points) } \lim_{x \rightarrow -2} \frac{x^3 - 4x}{x^2 - x - 6} = \lim_{x \rightarrow -2} \frac{x(x-2)(x+2)}{(x+2)(x-3)} \quad \left. \vphantom{\lim_{x \rightarrow -2} \frac{x(x-2)(x+2)}{(x+2)(x-3)}} \right\} 2 \text{ pts.}$$

$$= \lim_{x \rightarrow -2} \frac{x(x-2)}{x-3} = \frac{-2(-4)}{-5} = \boxed{-\frac{8}{5}} \quad \left. \vphantom{\lim_{x \rightarrow -2} \frac{x(x-2)}{x-3}} \right\} 2 \text{ pts.}$$

$$(c) \text{ (3 points) } \lim_{x \rightarrow \infty} \frac{2x^3 - 2x^2 + 4x - 1}{8 - 5x + 3x^2 - 3x^3}$$

$$= \boxed{-\frac{2}{3}} \quad \left. \vphantom{\lim_{x \rightarrow \infty} \frac{2x^3 - 2x^2 + 4x - 1}{8 - 5x + 3x^2 - 3x^3}} \right\} 3 \text{ pts.}$$

$$(d) \text{ (3 points) } \lim_{x \rightarrow 0^-} \frac{4x + |x|}{x}$$

$$= \lim_{x \rightarrow 0} \frac{4x - x}{x} = \lim_{x \rightarrow 0} \frac{3x}{x} = \boxed{3} \quad \left. \vphantom{\lim_{x \rightarrow 0} \frac{3x}{x}} \right\} 1 \text{ pt.}$$

$$(e) \text{ (3 points) } \lim_{x \rightarrow 3^-} \frac{2x - \lceil x \rceil}{x + \lceil x \rceil}, \text{ where } \lceil x \rceil \text{ is the greatest integer less than or equal to } x$$

$$= \frac{2(3) - 2}{3 + 2} = \boxed{\frac{4}{5}} \quad \left. \vphantom{\lim_{x \rightarrow 3^-} \frac{2x - \lceil x \rceil}{x + \lceil x \rceil}} \right\} 1 \text{ pt.}$$

3. Evaluate the limit if it exists. If it does not exist, explain why. Use the symbols ∞ or $-\infty$ as appropriate.

(a) (3 points) $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) = 0$

1 pt. by the squeeze theorem

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2 \quad \left. \vphantom{-x^2} \right\} 1 \text{ pt.}$$

↓

0 ← 4 pt.

(b) (5 points) $\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 9}}{2x + 1}$

3 pts

$$= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 \left(4 + \frac{9}{x^2}\right)}}{2x + 1}$$

$$= \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{4 + \frac{9}{x^2}}}{2x + 1} = \lim_{x \rightarrow -\infty} \frac{-x \sqrt{4 + \frac{9}{x^2}}}{2x + 1} =$$

$$= -\frac{1}{2} \cdot 2 = \boxed{-1} \quad 2 \text{ pts}$$

(c) (5 points) $\lim_{\theta \rightarrow 0} \frac{\sqrt{2\theta + 3} - \sqrt{3}}{\sin \theta}$

$$= \lim_{\theta \rightarrow 0} \frac{\sqrt{2\theta + 3} - \sqrt{3}}{\sin \theta} \cdot \frac{\sqrt{2\theta + 3} + \sqrt{3}}{\sqrt{2\theta + 3} + \sqrt{3}}$$

$$= \lim_{\theta \rightarrow 0} \frac{2\theta}{\sin \theta [\sqrt{2\theta + 3} + \sqrt{3}]} = \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{2\theta}} \cdot \lim_{\theta \rightarrow 0} \frac{1}{\sqrt{2\theta + 3} + \sqrt{3}}$$

$$= \frac{1}{\frac{1}{2}} \cdot \frac{1}{2\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}} \quad 2 \text{ pts}$$

(d) (5 points) $\lim_{x \rightarrow 0} \frac{x^2 - 1 + \cos^2 x}{x - \sin x}$

$$= \lim_{x \rightarrow 0} \frac{x^2 - 1 + 1 - \sin^2 x}{x - \sin x} \quad \left. \vphantom{x^2} \right\} 1 \text{ pt.}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x - \sin x} = \lim_{x \rightarrow 0} \frac{(x - \sin x)(x + \sin x)}{x - \sin x} \quad 2 \text{ pts}$$

$$= \lim_{x \rightarrow 0} (x + \sin x) = 0 \quad \left. \vphantom{x} \right\} 2 \text{ pts}$$

4. (6 points) Let

$$f(x) = \begin{cases} ax + 10 & \text{if } x \leq -2 \\ b|x| & \text{if } -2 < x \leq 2 \\ a(x-2)^2 + 6 & \text{if } x > 2 \end{cases}$$

Find the values of the constants a and b that make $f(x)$ continuous on $(-\infty, \infty)$.

① $f(x)$ is cont. at $x = -2 \Rightarrow$

1 pt \leftarrow

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^-} f(x) \Rightarrow$$

$$b|-2| = a(-2) + 10$$

1 pt \leftarrow

$$2b = -2a + 10 \quad \dots \text{I}$$

② $f(x)$ is cont. at $x = 2 \Rightarrow$

1 pt \leftarrow

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x)$$

1 pt \leftarrow

$$6 = b|-2| \quad \dots \text{II}$$

\Rightarrow

$$6 = 2b \Rightarrow \boxed{b = 3} \quad \left. \vphantom{6 = 2b} \right\} 1 \text{ pt.}$$

$$2(3) = -2a + 10$$

$$6 = -2a + 10$$

$$\frac{6-10}{-2} = a \Rightarrow$$

$$\boxed{a = 2} \quad \left. \vphantom{a = 2} \right\} 1 \text{ pt.}$$

5. (8 points) Use the **definition** of the derivative to find $f'(a)$ for $f(x) = \sqrt{5-3x}$.

$$\frac{2 \text{ pts.}}{\swarrow} \left\{ f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \right.$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{5-3(a+h)} - \sqrt{5-3a}}{h}$$

$$\frac{2 \text{ pts.}}{\swarrow} \left\{ = \lim_{h \rightarrow 0} \frac{\sqrt{5-3(a+h)} - \sqrt{5-3a}}{h} \cdot \frac{\sqrt{5-3(a+h)} + \sqrt{5-3a}}{\sqrt{5-3(a+h)} + \sqrt{5-3a}} \right.$$

$$= \lim_{h \rightarrow 0} \frac{5-3(a+h) - (5-3a)}{h \left[\sqrt{5-3(a+h)} + \sqrt{5-3a} \right]}$$

$$= \lim_{h \rightarrow 0} \frac{5-3a-3h-5+3a}{h \left[\sqrt{5-3(a+h)} + \sqrt{5-3a} \right]}$$

$$= \lim_{h \rightarrow 0} \frac{-3h}{h \left[\sqrt{5-3(a+h)} + \sqrt{5-3a} \right]}$$

$$= \lim_{h \rightarrow 0} \frac{-3}{\sqrt{5-3(a+h)} + \sqrt{5-3a}}$$

$$= \frac{-3}{\sqrt{5-3a} + \sqrt{5-3a}}$$

$$= \boxed{\frac{-3}{2\sqrt{5-3a}}} \left. \begin{array}{l} \text{2 pts.} \\ \nearrow \end{array} \right\}$$

6. The displacement (in meters) of a particle moving in a straight line is given by $s(t) = 3t^2 + 1$, where t is measured in seconds and $0 \leq t \leq 5$.

(a) (4 points) Find the average velocity over the interval $[0, 5]$.

$$\begin{aligned} V_{\text{ave}} &= \frac{s(5) - s(0)}{5 - 0} \quad \left. \vphantom{\frac{s(5) - s(0)}{5 - 0}} \right\} \xrightarrow{2 \text{ pts}} \\ &= \frac{76 - 1}{5} = \underline{15} \text{ m/s} \\ &\quad \left. \vphantom{\frac{76 - 1}{5}} \right\} \xrightarrow{2 \text{ pts.}} \end{aligned}$$

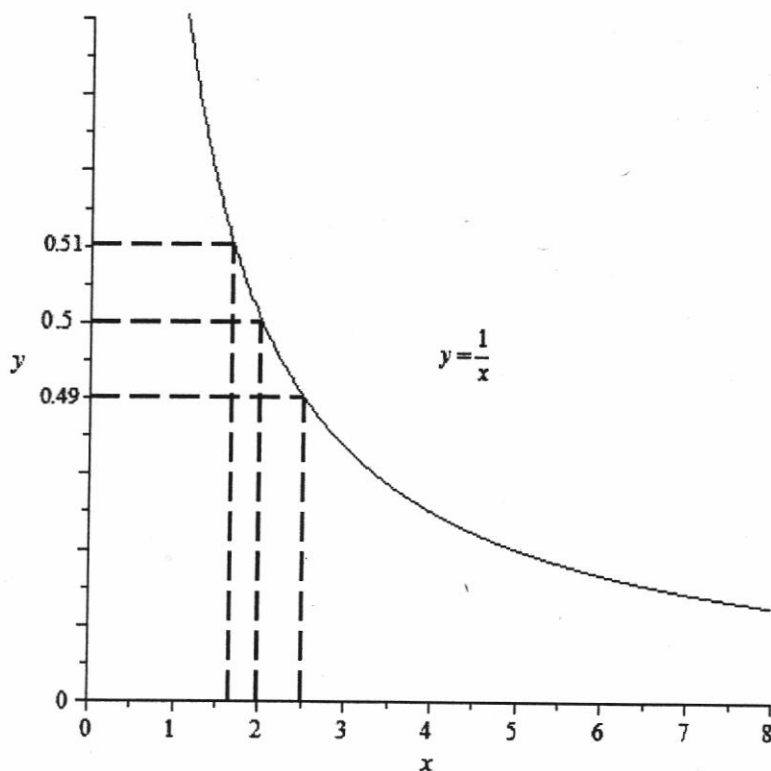
(b) (4 points) Use limits to find the instantaneous velocity v when $t = 4$.

$$\begin{aligned} v_{\text{ins}}(4) &= \lim_{t \rightarrow 4} \frac{s(t) - s(4)}{t - 4} \quad \left. \vphantom{\lim_{t \rightarrow 4} \frac{s(t) - s(4)}{t - 4}} \right\} \xrightarrow{2 \text{ pts.}} \\ &= \lim_{t \rightarrow 4} \frac{3t^2 + 1 - 49}{t - 4} \\ &= \lim_{t \rightarrow 4} \frac{3t^2 - 48}{t - 4} \\ &= \lim_{t \rightarrow 4} \frac{3(t-4)(t+4)}{(t-4)} \quad \left. \vphantom{\lim_{t \rightarrow 4} \frac{3(t-4)(t+4)}{(t-4)}} \right\} \xrightarrow{1 \text{ pt.}} \\ &= \lim_{t \rightarrow 4} 3(t+4) = 24 \quad \left. \vphantom{\lim_{t \rightarrow 4} 3(t+4)} \right\} \xrightarrow{1 \text{ pt.}} \end{aligned}$$

7. (6 points) Use the given graph of $f(x) = \frac{1}{x}$ to find a number δ such that

$$|f(x) - 0.5| < 0.01 \text{ whenever } 0 < |x - 2| < \delta.$$

Write your answer in the form $\delta = \frac{p}{q}$ where p and q are integers.



$$f(x) = \frac{1}{x} \Rightarrow$$

$$y_1 = 0.51 \Rightarrow \frac{1}{x_1} = 0.51 \Rightarrow x_1 = \frac{100}{51} \quad \xrightarrow{1 \text{ pt.}}$$

$$y_2 = 0.49 \Rightarrow \frac{1}{x_2} = 0.49 \Rightarrow x_2 = \frac{100}{49} \quad \xrightarrow{1 \text{ pt.}}$$

$$\delta = \min \left\{ 2 - \frac{100}{51}, \frac{100}{49} - 2 \right\} \quad \xrightarrow{3 \text{ pts.}}$$

$$= \min \left\{ \frac{2}{51}, \frac{2}{49} \right\}$$

$$= \frac{2}{51} \quad \xrightarrow{1 \text{ pt.}}$$

8. Consider $f(x) = \frac{x}{2x^2 + x}$.

(a) (2 points) Find all values of x where f is **not** continuous.

$f(x) = \frac{x}{x(2x+1)} \Rightarrow f(x)$ is NOT cont.

at

① $x = 0$

1 pt.

② $x = -1/2$

1 pt.

(b) (4 points) Evaluate the limit of f at each of these values, and state the type of discontinuity at each point.

① $\lim_{x \rightarrow 0} \frac{x}{x(2x+1)} = \lim_{x \rightarrow 0} \frac{1}{2x+1} = 1$ 1 pt.
 "Removable" 1 pt.

② $\lim_{x \rightarrow -1/2} \frac{x}{x(2x+1)} = \text{does not exist.}$ 1 pt.
 "Infinite" 1 pt.

9. (6 points) Find all the horizontal and vertical asymptotes of $f(x) = \frac{x}{\sqrt{x^2 + 1}}$.

① Vertical Asymptotes: None } 2 pts.

② Horizontal Asymptotes

(i) $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow \infty} \frac{x}{|x| \sqrt{1 + \frac{1}{x^2}}} = 1$ 1 pt.
 \Rightarrow $y = 1$ 1 pt.

(ii) $\lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow -\infty} \frac{x}{|x| \sqrt{1 + \frac{1}{x^2}}} = -1$ 1 pt.
 \Rightarrow $y = -1$ 1 pt.

10. (10 points) Suppose $f(x) = \frac{x}{x-3}$. If $f'(a) = \frac{-3}{(a-3)^2}$, Find the point (a, b) on the graph of $f(x)$ such that the tangent line to the graph at (a, b) passes through the point $(9, 1)$.

Solution:

The equation of the tangent line at (a, b) is

$$y - y_1 = m(x - x_1)$$

$$\text{where } m = f'(a) = \frac{-3}{(a-3)^2} \quad \left. \vphantom{m} \right\} \text{ 2 pts.}$$

$$x_1 = a, \quad y_1 = b \Rightarrow$$

$$y - b = \frac{-3}{(a-3)^2}(x - a) \quad \left. \vphantom{y} \right\} \text{ 2 pts.}$$

Now: since (a, b) is on the graph of

$$f(x), \text{ then } b = f(a) = \frac{a}{a-3} \Rightarrow$$

$$y - \frac{a}{a-3} = \frac{-3}{(a-3)^2}(x - a) \quad \left. \vphantom{y} \right\} \text{ 2 pts.}$$

Also, the line goes through $(9, 1)$ and so

$$1 - \frac{a}{a-3} = \frac{-3}{(a-3)^2}(9 - a) \quad \left. \vphantom{1} \right\} \text{ 2 pts.}$$

$$\Rightarrow (a-3)^2 - a(a-3) = -3(9-a)$$

$$\Rightarrow a^2 - 6a + 9 - a^2 + 3a = -27 + 3a$$

$$\Rightarrow -6a = -36 \Rightarrow \boxed{a = 6} \quad \left. \vphantom{a} \right\} \text{ 1 pt.}$$

$$\text{and so } b = \frac{a}{a-3} = \frac{6}{6-3} = 3$$

$$\Rightarrow \boxed{b = 3} \quad \left. \vphantom{b} \right\} \text{ 1 pt.}$$

11. (6 points) Use the Intermediate Value Theorem to show that the functions $f(x) = \ln(x)$ and $g(x) = e^{-x}$ intersect over the interval $[1, e]$.

Solution:

$$\begin{aligned} \text{Let } h(x) &= f(x) - g(x) \\ &= \ln x - e^{-x} \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{Let } h(x) &= f(x) - g(x) \\ &= \ln x - e^{-x} \end{aligned}} \right\} \xrightarrow{2 \text{ pts}}$$

Then

$\xleftarrow{1 \text{ pt.}}$ ① $h(x)$ is cont. (both f & g are cont.)

$\xleftarrow{1 \text{ pt.}}$ ② $h(1) = \ln 1 - e^{-1} = -\frac{1}{e} < 0$

$\xleftarrow{1 \text{ pt.}}$ ③ $h(e) = \ln e - e^{-e}$
 $= 1 - \frac{1}{e^e} > 0$

[note $e^e > 1 \Rightarrow \frac{1}{e^e} < 1$]

by the IMVT, there exists a number c such that

$$1 < c < e$$

and

$$h(c) = 0 \Rightarrow$$

$$\ln c - e^{-c} = 0$$

$$\Rightarrow \boxed{\ln c = e^{-c}}$$

} 1 pt