11.8 Power Series

A series of the form

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$ where x is variable and $c'_n s$ are constants.

If the power series converges for some x, it's sum is a function

Sum= $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ... + c_n x^n + ... = \sum_{n=0}^{\infty} c_n x^n$

Domain: set of all x where it converges.

Example: $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ...$ where domain

$|x| < 1$ (called the radius of convergence) and diverges $|x| \geq 1$

Power series centered about *a* or in *(x – a)* or power series about a

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + ...$

If x = a it converges to c_0 .

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are three possibilities

(i) The series converges only when x =a
(ii) The series converges for all x
(iii) There is a positive number R such that series converges if |x-a|<R and diverges if |x-a|>R. **R is called RADIUS OF CONVERGENCE**

Interval of convergence (I) is the interval where the series is convergent.
If it converges at ONE point say b then R=0 and I ={b}
If it converges at for all x then R = ∞ and I = $(-\infty, \infty)$
If it converges for |x-a|<R then Radius is R and Interval is (a - R, a + R). We have to test at the end points at a-R and a +R to find the INTERVAL I

<u>Finding values of x where a given series converges:</u>

Example 1(book): For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ is convergent

Apply Ratio Test, $x \neq 0$, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

By Ratio Test, the series diverges when $x \neq 0$. It is convergent if x = 0.
R=0 and Interval of convergence is {0}

Example 2(book) : For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ is convergent.

Apply Ratio Test, have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+1}.\frac{n}{(x-3)^n}\right| = |x-3|$$

By Ratio Test

- It is convergent if |x-3|<1
- It is divergent if |x-3|>1
- The test fails if | x - 3| = 1. We have test for x – 3 = 1 and x - 3 = -1 → x = 4 and x = 2

If x = 4 the series becomes $\sum \frac{1}{n}$ (Divergent)

If x = 2 the series becomes $\sum \frac{(-1)^n}{n}$ (Convergent, by Alternating test)

Given series is convergent $2 \le x < 4$

R=1 and Interval of convergence is [2, 4)

<u>READ EXAMPLE 4, 5, and 6.</u>

Recitation Problems

Ex1: (book ex-8) $\sum_{n=1}^{\infty} n^n x^n$

We can apply Root Test $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} n|x| = \infty$ if $x \neq 0$

If x = 0 , it converges , R = 0, Interval of convergence = {0}

Ex2: (book ex-20) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n\ 3^n}$

We can apply Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{(n+1)3^{n+1}}.\frac{n3^n}{(3x-2)^n}\right| = \lim_{n\to\infty}\frac{|x-3|}{3}\frac{n}{n+1} = \frac{|3x-2|}{3} = \left|x-\frac{2}{3}\right|$$

Converges if $\left|x-\frac{2}{3}\right| < 1$ → R=1 and we can find interval $-1 < x - \frac{2}{3} < 1 ==> \frac{-1}{3} < x < \frac{5}{3}$

When x = -1/3 , The series becomes $\sum \frac{(-1)^n}{n}$ Alternating (Convergent)

When x = 5/3 , The series becomes $\sum \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ (Divergent)

Interval of Convergence $\frac{-1}{3} \le x < \frac{5}{3}$

Ex3: (book ex-29) If $\sum_{n=0}^{\infty} c_n 4^n$ is convergent, does it follow that following series are convergent (a) $\sum_{n=0}^{\infty} c_n (-2)^n$ (b) $\sum_{n=0}^{\infty} c_n (-4)^n$

(a) Given that $\sum_{n=0}^{\infty} c_n 4^n$ is convergent ➔ it is convergent for x = 4. By Theorem 3, it must converge at least in $-4 < x \le 4$ ➔ It must converge at x = -2
→ $\sum_{n=0}^{\infty} c_n (-2)^n$ is convergent.

(b) It is not necessarily convergent at the other end point x = -4. That may not be convergent

Ex4: (book ex-21) Find the radius of convergence and the interval of convergence for $\sum_{n=1}^{\infty} \frac{n(x-a)^n}{b^n}, b > 0$

Apply Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-a)^{n+1}}{b^{n+1}}.\frac{b^n}{n(x-a)^n}\right| = \lim_{n\to\infty}\frac{|x-a|}{b}\frac{(1+1/n)}{} = \frac{|x-a|}{b} \quad \text{(b>0)}$$

The series converges if $\frac{|x-a|}{b} < 1$ ➔ **| x-a |< b➔ a - b< x < a + b**

We need to test when | x-a |= b, The series becomes $\sum_{n=1}^{\infty} n$ ➔ Divergent