11.5 Alternating Series

Summary: An alternating series is a series whose terms are alternatively positive and negative.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+ \qquad = \sum_{i=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

The n^{th} term is $a_n=(-1)^{n-1}b_n$ or $(-1)^n b_n$ where $b_n=|a_n|$.

CONVERGENCE TEST:

Given the alternating series as $\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+.....$ where $b_n>0$ **is said to be Convergent if it satisfies two conditions**

1. $b_{n+1}\le b_n$ for all n (some terms may be excluded) [DECREASING]
2. The limit of nth term tends to zero $\lim_{n->\infty} b_n=0$

Estimating Sums: The partial sum s_n of any **convergent can be used as an approximation to the total sum s (** $s\simeq s_n$). For this we must find the remainder $R_n=s-s_n$ where $|R_n|=|s-s_n|\le b_{n+1}$

Ex1:(book-16). Test the convergence of $\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n!}$

To convergence test of alternating series we have to check if the series is alternating. We see that

(a) The terms of the series are zero **when n is even** $\sin\frac{n\pi}{2}=0$. It means it has no effect on the series. We can exclude such terms. We are left with odd terms

(b) If n is odd n=2k+1, then $\sin\frac{(2k+1)\pi}{2}=(-1)^k$. And the series becomes $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1!}$.

The series is alternating with $b_k=\frac{1}{2k+1}>0$ and

1. It is decreasing ($b_{k+1}<b_k$)
2. $\lim_{k->\infty} b_k=0$

➔ CONVERGENT

Ex2(book-17). Test the convergence of $\sum_{n=1}^{\infty}(-1)^n\sin\frac{\pi}{n}$**. We see that for all n=1, 2, the** $\sin\frac{\pi}{n}$ **is in First or Second Quadrant and is positive for n>/2. Also**

1. It is decreasing .. bigger the value of n smaller is value of **sin (** $b_{n+1} \le b_n$)

2. $\lim_{n->\infty} b_n = \lim_{n->\infty} \sin\left(\frac{\pi}{n}\right) = 0$

➔ CONVERGENT

Ex3(book-28). Approximate the sum of the series correct to 4 decimal places

$\sum_{n=1}^{\infty}(-1)^n \frac{n}{8^n}$

➔Add many terms that gives sum up to 5 places and fourth place is more than 5

$s = -\frac{1}{8} + \frac{2}{8^2} - \frac{3}{8^3} + \frac{4}{8^4} - \frac{5}{8^5} + \frac{6}{8^6}$ the 6th term is $b_6 = 0.000023$

Sum up to 5 terms $s \simeq s_5 = -0.098785$. Adding 6th term 0.000023 does not affect the fourth decimal place o the sum. **Therefore sum = -0.0988**

Ex4(book-34). **Test the convergence of** $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(\ln n)^p}{n}$**.**

The series is alternating wiht $b_k = \frac{(\ln k)^p}{k}$

1. Check if the series is decreasing: We can write

$f(x) = \frac{(\ln x)^p}{x}$ *and* $f^{'}(x) = \frac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ If (p –ln x) is < 0 ➔ $x > e^p$. It means we can find x (depending on p) where the series will start decreasing.

2. We have to show $\lim_{k->\infty} b_k = 0$

(a) If $p \le 0$, $\lim_{n->\infty} b_k = \lim_{k->\infty} \frac{|\ln k|^p}{k}$ will be 0, since ln k will be in the numerator.

(b) If $p > 0$. We can apply Hospital Rule and differentiate the numerator and denominator as many as times as power of ln x is positive. Once it becomes negative, the ln x will go to numerator and the limit will become 0.

$$\lim_{x->\infty} f(x) = \lim_{x->\infty} \frac{(\ln x)^p}{x}$$

$$= \lim_{x->\infty} \frac{p(\ln x)^{p-1}}{x}$$

$$= \lim_{x->\infty} \frac{p(p-1)(\ln x)^{p-2}}{x} = \lim_{x->\infty} \frac{p(p-1)(p-2)(\ln x)^{p-2}}{x}$$

Finally ln x wil go in denominator and the limit will be zero.

➔ CONVERGENT