Asymptotic Behavior of a Controlled Branching Process with Continuous State Space

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ABSTRACT

In the paper a modification of the branching stochastic process with immigration and with continuous states, introduced by Adke and Gadag [1] will be considered. Theorems establishing a relationship of this process with $Bienaymé - Galton - Watson$ processes will be proved. It will be demonstrated that limit theorems for the new process can be deduced from those for simple processes with time-dependent immigration, assuming that process is critical and offspring variance is finite.

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1. INTRODUCTION

We consider a modification of the branching stochastic process which has a continuous space of states. It is convenient to define the process as a family of nonnegative random variables describing the amount of a product produced by individuals of some population. The initial state of the process is given by a nonnegative random variable X_0 . The amount of the product X_1 of the first generation is defined as the sum of random products produced by $N_1(X_0)$ individuals and the product U_1 of immigrating to the first generation individuals. Similarly the amount X_2 of the product of the second generation is defined as the sum of products produced by $N_2(X_1)$ individuals and U_2 , and so on. Here $N_k(t), k \geq 1, t \in T$, are counting processes with independent stationary increments, T is either $R_+ = [0, \infty)$ or $Z_+ = \{0, 1, 2, ...\}$ and $U_k, k \geq 1$, are non-negative random variables. We also assume that processes $N_n(t), n \geq 1, t \in T$ have common one dimensional distributions. This process allow to model situations, when it is difficult to count the number of individuals in the population, but some non-negative characteristic, such as volume, weight or product produced by the individuals can be measured.This modification of branching processes was introduced by Adke and Gadag (1995), who indicated relationship of this model with problems related to non-Gaussian Markov time series, to single server queue models and to other problems.

Investigation of branching processes with continuous state space has a long history. This kind processes were first introduced by Feller [5] who studied a class of one dimensional diffusions obtained by a passage to the limit from the $Bienaym\acute{e} - Galton - Watson$ (BGW) processes. At the end of fifties M. Jirina [9], [10] defined a branching stochastic process with continuous-state space as a homogeneous Markov process the transition probabilities of which satisfy a "branching condition". The continuous-state branching process with immigration was considered by Kawazu and Watanabe [11]. Since then investigations of various models of the branching process with continuous states have been active area of the research. We just note most recent publications by Zeng [16], Lambert [12] and Duquesne [4], where genealogical trees associated with continuous-state branching processes are considered. Additional references in this direction can be found in Athreya and Ney [2].

In the case, when X_0 and $U_k, k \geq 1$, are integer-valued, the process X_n can be considered as a special case of a controlled branching process introduced first by Sevastyanov and Zubkov [14] and by Yanev [15] , for random control functions. In fact, if we choose $\varphi_1(k,n) = N_k(n)$ and $\varphi_2(k,n) \equiv 1$ in so called Model 2 of φ- branching process [14], obtain a discrete-state version of the process X_n. Further investigations of controlled branching processes with random control functions can be found in [8] and the references wherein.

As distinct from the cited above papers, where the process has been given by a special form of the Laplace transform, in the process which we are going to consider the branching property can explicitly be presented using counting process $N_n(t)$. This allowed Adke and Gadag [1] to obtain distributional properties of the process X_n that are similar to those of classic models. In particular it was shown that $Z_n = N_{n+1}(X_n)$ is simple BGW process with time-dependent immigration. The following question is interesting in connection with this situation. Is it possible to use this similarity in investigation of asymptotic behavior of the process? In particular can we obtain limit distributions of X_n directly from known limit theorems for BGW processes?

In this paper we prove certain theorems which establish relationship between these two processes in a sense of asymptotic behavior. These results allow us to prove limit theorems for X_n from those of Z_n and vice versa. We demonstrate usefulness of these theorems obtaining limit distributions for the critical process with time-dependent immigration in cases of linear and functional normalization. New limit theorems for critical processes X_n with finite variance of offspring distribution will be proved when immigration rate decreases depending the time of immigration and also when it satisfies Foster-Williamson condition of weak stability. In further publications we will demonstrate applicability of these duality theorems when offspring variance is not finite, to subcritical and supercritical processes and to processes without immigration.

Hence considered here continuous-state process can be treated by traditional for the theory of branching processes technique, while it may serve to model continuously varying branching populations as the more complicated Jirina or Kawazu-Watanabe processes.

2. TWO DUALITY RESULTS

We now give a detailed definition of the process which we are going to consider. Let $\{W_{in}, i, n \geq 1\}$ be a double array of independent and identically distributed non-negative random variables, $\{N_n(t), t \in T, n \geq 1\}$ be a family of nonnegative, integer-valued independent processes with independent stationary increments, with $N_n(0) = 0$ almost surely, T is either $R_+ = [0, \infty)$ or $Z_+ = \{0, 1, ...\}$.

We define a new process $X_n, n \geq 0$, as following. Let the initial state of the process be X_0 which is an arbitrary non-negative random variable and for $n \geq 0$

$$X_{n+1} = \sum_{i=1}^{N_{n+1}(X_n)} W_{in+1} + U_{n+1}, \tag{1}$$

where $\{U_n, n \geq 1\}$ is a sequence of independent non-negative random variables. Assume that families of random variables $\{W_{in}, i, n \geq 1\}, \{U_n, n \geq 1\}$ of stochastic processes $\{N_n(t), t \in T, n \geq 1\}$ and random variable X_0 are independent.

It is shown in [1] that $Z_n = N_n(X_{n-1})$ is a BGW process with an immigration component. We now provide a result establishing relationship, in a sense of limiting behavior, between processes X_n and Z_n. In order to do that we use the following Laplace transforms

$$G(\lambda) = Ee^{-\lambda W_{in}}, H_n(\lambda) = Ee^{-\lambda U_n}.$$

We also denote

$$\Delta(n) = \frac{P\{Z_n > 0\}}{P\{X_n > 0\}}, \delta(n, \lambda) = \frac{1 - H_n(\lambda)}{P\{Z_n > 0\}}.$$

Let the sequences of positive numbers $\{k(n), n \geq 1\}$ and $\{a(n), n \geq 1\}$ be such that $k(n), a(n) \to \infty$ and for each $\lambda > 0$ there exists

$$\lim_{n\to\infty} k(n)(1 - G(\frac{\lambda}{a(n)})) = b(\lambda) \in (0, \infty). \tag{2}$$

Existence of these sequences follows from monotonicity of the Laplace transform $G(\lambda)$. In fact one may choose

$$a(n) = \frac{\lambda}{G^{-1}(1 - \frac{b(\lambda)}{k(n)})}$$

for a given sequence $k(n)$, where G^{-1} stands for the inverse of $G(\lambda)$.

Theorem 1. *Let $\Delta(n) \to 1, n \to \infty$ and $\delta(n, \lambda/a(n)) \to 0$ for each $\lambda > 0$ as $n \to \infty$. Then as $n \to \infty$*

$$E[e^{-\lambda X_n/a(n)}|X_n > 0] \to \phi(b(\lambda)) \tag{3}$$

for each $\lambda > 0$, if and only if as $n \to \infty$ for each $\lambda > 0$

$$E[e^{\lambda Z_n/k(n)}|Z_n > 0] \to \phi(\lambda). \tag{4}$$

Proof. We consider the following obvious identity

$$E[e^{-\lambda X_n}|X_n > 0] = 1 - \frac{1 - Ee^{-\lambda X_n}}{P\{X_n > 0\}}. \tag{5}$$

It follows from definition (1) of the process X_n by total probability arguments that

$$Ee^{-\lambda X_n} = H_n(\lambda) EG^{Z_n}(\lambda). \tag{6}$$

We obtain from (6) that

$$1 - Ee^{-\lambda X_n} = (1 - H_n(\lambda))EG^{Z_n}(\lambda) + 1 - EG^{Z_n}(\lambda).$$

Thus making use of (5)

$$\frac{1 - Ee^{-\lambda X_n}}{P\{Z_n > 0\}} = 1 - E[G^{Z_n}(\lambda)|Z_n > 0] + \delta(n, \lambda)E[G^{Z_n}(\lambda)].$$

Hence the ratio on the right side of (5) equals

$$\Delta(n)\frac{1 - Ee^{-\lambda X_n}}{P\{Z_n > 0\}} = -\Delta(n)E[G^{Z_n}(\lambda)|Z_n > 0] + \Delta(n)[1 + \delta(n, \lambda)EG^{Z_n}(\lambda)].$$

If we use this in relation (5) we obtain

$$E[e^{-\lambda X_n}|X_n > 0] = \Delta(n)E[G^{Z_n}(\lambda)|Z_n > 0] + \varepsilon(n), \tag{7}$$

where

$$\varepsilon(n) = 1 - \Delta(n)(1 + \delta(n, \lambda))E[G^{Z_n}(\lambda)].$$

Let (4) be satisfied for every $\lambda > 0$. Then, it clearly follows from continuity of the Laplace transform $\varphi(\lambda)$, that the convergence in (4) holds uniformly with respect to λ in an arbitrary finite interval. Since $\ln x = -(1-x) + o(1-x), x \to 1$, we obtain from condition (2) that as $n \to \infty$

$$t_n = -k(n) \ln G(\frac{\lambda}{a(n)}) \to b(\lambda). \tag{8}$$

Therefore for each fixed $\lambda > 0$ there is such a $T = T(\lambda)$, that $0 < t_n \leq T$ for any $n = 1, 2, ...$ Replacing λ by $\lambda/a(n)$ and using (8) we have

$$E[G^{Z_n}(\frac{\lambda}{a(n)})|Z_n > 0] = E[e^{-t_n Z_n/k(n)}|Z_n > 0]. \tag{9}$$

We show that the Laplace transform (9) as $n \to \infty$ approaches $\varphi(b(\lambda))$. In order to do this we consider the following relation:

$$E[G^{Z_n}(\frac{\lambda}{a(n)})|Z_n > 0] - \varphi(b(\lambda)) = I_1 + I_2, \tag{10}$$

where

$$I_1 = E[e^{-t_n Z_n/k(n)}|Z_n > 0] - \varphi(t_n), I_2 = \varphi(t_n) - \varphi(b(\lambda)).$$

It follows from (4), due to the uniform convergence, that

$$|I_1| \leq \sup_{0<t_n<T} |E[e^{-t_n Z_n/k(n)}|Z_n > 0] - \varphi(t_n)| \to 0 \tag{11}$$

as $n \to \infty$. On the other hand $I_2 \to 0$ as $n \to \infty$ due to continuity of the Laplace transform $\varphi(\lambda)$, for $\lambda > 0$. Thus we conclude that as $n \to \infty$

$$E[G^{Z_n}(\frac{\lambda}{a(n)})|Z_n > 0] \to \varphi(b(\lambda)). \tag{12}$$

Since $\Delta(n) \to 1$ and $\delta(n, \frac{\lambda}{a(n)}) \to 0$ as $n \to \infty$, we obtain that $\varepsilon(n) \to 0$ as $n \to \infty$. The assertion (3) now follows from relations (7) and (12). The first part of Theorem 1 is proved.

Let now (3) hold. Recall that $t_n = -k(n) \ln G(\frac{\lambda}{a(n)})$. It follows from condition (2) that $\tau_n = t_n/b(\lambda) \to 1$ as $n \to \infty$ for each $\lambda > 0$. We consider the following Laplace transform:

$$E[e^{-Z_n b(\lambda) \tau_n/k(n)}|Z_n > 0] = E[G^{Z_n}(\frac{\lambda}{a(n)})|Z_n > 0]. \tag{13}$$

It follows from relations (3), (7) and (13), due to continuity of $\varphi(\lambda)$, that

$$\lim_{n\to\infty} E[e^{-Z_n b(\lambda)\tau_n/k(n)}|Z_n > 0] = \varphi(b(\lambda)). \tag{14}$$

Due to continuity theorem for Laplace transforms (14) means that

$$\left\{\frac{Z_n \tau_n}{k(n)}|Z_n > 0\right\} \xrightarrow{\mathrm{D}} \xi$$

as $n \to \infty$, with $Ee^{-\lambda\xi} = \varphi(\lambda)$. Since $\tau_n \to 1, n \to \infty$, we have that $Z_n/k(n)$ given $Z_n > 0$, as $n \to \infty$ converges to ξ in distribution. If we write this in terms of Laplace transforms, we obtain (4). Theorem 1 is proved completely.

Now we obtain a similar duality result for unconditional distributions of processes Z_n and X_n. It will also be formulated it terms of Laplace transforms.

Theorem 2. *Let for sequences $\{a(n), n \geq 1\}$ and $\{k(n), n \geq 1\}$ condition (2) be satisfied. Then*

$$Ee^{-\lambda X_n/a(n)} \to \varphi(b(\lambda)) \tag{15}$$

if and only if for each $\lambda > 0$ as $n \to \infty$

$$Ee^{-\lambda Z_n/k(n)} \to \varphi(\lambda). \tag{16}$$

Proof. Now we use equation (6) directly. Let (16) be satisfied for each $\lambda > 0$. Then it holds uniformly with respect to $\lambda > 0$ from each finite interval. Again taking into account relation (9) we can partition $E[G^{Z_n}(\lambda/a(n))] - \varphi(b(\lambda))$ into $I_1 + I_2$ and, as in the proof of Theorem 1, show that both I_1 and I_2 approach zero as $n \to \infty$. This leads assertion of (15) due to relation (6).

The proof of the necessity of (16) for (15) is similar to the proof of the second part of previous theorem. One just needs to consider unconditional Laplace transforms instead of conditional ones. Theorem 2 is proved.

Now we turn our attention to some applications of proved theorems. In order to do it we need explicit formulas for moments of offspring and immigration distributions of the process Z_n.

3. OFFSPRING AND IMMIGRATION MOMENTS

As it was indicated before process $Z_n = N_n(X_{n-1})$ is a BGW process with immigration. The offspring distribution and the distribution of the number of immigrating masses have Laplace transforms $G(f(\lambda)) = Ee^{-\lambda \xi_n}$ and $H_n(f(\lambda)) = Ee^{-\lambda \eta_n}$, respectively (see [1]). Here $\xi_n = N_n(W_{n-1}), \eta_n = N_n(U_{n-1})$ and $f(\lambda) = -\log Ee^{-\lambda N_n(1)}$.

We obtain the moments of offspring and immigration distributions by standard arguments. It is easy to see that

$$m = E\xi_n = -\frac{d}{d\lambda} G(f(\lambda))_{\lambda=0} = EWEN,$$

where $N = N_1(1), W = W_1$. Similarly

$$\alpha(n) = E\eta_n = -\frac{d}{d\lambda} H_n(f(\lambda))_{\lambda=0} = EU_nEN.$$

Since

$$\frac{d^2}{df^2} G(f(\lambda)) = \frac{d^2}{df^2} G(f(\lambda)) \left\{ \frac{df(\lambda)}{d\lambda} \right\}^2 + \frac{d}{df} G(f(\lambda)) \frac{d^2 f(\lambda)}{d\lambda^2},$$

we obtain

$$E\xi_n^2 = \left. \frac{d^2 G(f(\lambda))}{d\lambda^2} \right|_{\lambda=0} = EW^2(EN)^2 + EWVarN$$

One of the important parameters in the theory of usual branching processes is the factorial moment of the offspring distribution $B = E\xi_n(\xi_n - 1)$. We obtain from the above that

$$B = EW[VarN - EN] + EW^2(EN)^2.$$

In particular when $E\xi_n = 1$ (the critical case) we have

$$B = EWVarN + (EN)^2 VarW.$$

By similar arguments we obtain that

$$E\eta_n^2 = EU_nVarN + EU_n^2(EN)^2$$

and for the factorial moment $\beta(n) = E\eta_n(\eta_n - 1)$ we have

$$\beta(n) = (EN)^2 EU_n(U_n - 1) + EN(N - 1)EU_n$$

4. A FOSTER-WILLIAMSON TYPE THEOREM

Here we consider applicability of Theorem 2 to obtain a version of well known result by Foster and Williamson (1971). They assume convergence in distribution of the normalized immigration process (the partial sum of the number of immigrating individuals) to a random variable ξ. Since ξ is nonnegative and has an infinitely divisible distribution its Laplace transform has the form (see Feller [6], page 426)

$$Ee^{-\lambda\xi} = \exp\left\{-\int_0^\infty \frac{1-e^{-\lambda x}}{x} dP(x)\right\},$$

where $P(x)$ is a measure such that $\int_0^\infty x^{-1} dP(x) < \infty$. First we state the theorem for the process Z_n from [7].

Theorem A. *If $m = 1, B \in (0, \infty)$ and*

$$\frac{1}{n}\sum_{k=1}^{n} N_k(U_{k-1}) \xrightarrow{\text{D}} \xi, \tag{17}$$

then $Z_n/n \xrightarrow{\text{D}} W$, with

$$Ee^{\lambda W} = \exp\left\{-\int_0^\infty \frac{1-e^{-\lambda x}}{x} dQ(x)\right\},$$

where $Q(x) = R * P(x), R(x) = 1 - \exp\{-2x/B\}$.

Now we formulate Foster-Williamson type result for process X_n. It is natural that the condition on immigration must be given in terms of $\{U_k, k \geq 1\}$ the "immigrating mass".

Theorem 3. *If $m = 1, B \in (0, \infty)$ and*

$$\frac{EN}{n}\sum_{k=1}^{n} U_k \xrightarrow{\text{D}} \xi, \tag{18}$$

then $X_n/n \xrightarrow{\text{D}} X$, with

$$Ee^{\lambda X} = \exp\left\{-\int_0^\infty \frac{1-e^{-\lambda x EW}}{x} dQ(x)\right\},$$

and $Q(x)$ is the same as in Theorem A.

Proof First we show that, if condition (18) is fulfilled, then (17) holds. In fact, in terms of Laplace transforms (18) is

$$\prod_{k=1}^{n} H_k(\frac{\lambda EN}{n}) \to Ee^{-\lambda\xi}. \tag{19}$$

If we denote the sum in (17) by S_n, we have

$$Ee^{-\lambda S_n/n} = \prod_{k=1}^{n} H_k(f(\frac{\lambda}{n})).$$

Using relation $\log x = -(1-x) + o(1-x), x \downarrow 1$, we obtain that

$$nf(\frac{\lambda}{n}) = -n \log Ee^{-\lambda N/n} \sim n(1 - Ee^{-\lambda N/n}) \sim \lambda EN$$

as $n \to \infty$. Thus, due to continuity of the Laplace transform, we conclude that

$$\prod_{k=1}^{n} H_k(f(\frac{\lambda}{n})) \sim \prod_{k=1}^{n} H_k(\frac{\lambda EN}{n})$$

and this together with (19) gives (17).

It follows from the above that, if condition (18) is satisfied, then Theorem A holds, i. e. $Z_n/n \xrightarrow{\mathrm{D}} W, n \to \infty$. This can be written in terms of Laplace transforms as $Ee^{-\lambda Z_n/n} \to Ee^{-\lambda W}, n \to \infty$. Now we appeal to Theorem 2. If we choose $k(n) = a(n) = n$, then as $n \to \infty$

$$n(1 - G(\frac{\lambda}{n})) \to \lambda EW.$$

Thus condition (2) is fulfilled with $b(\lambda) = \lambda EW$. The assertion of Theorem 3 now follows from Theorem 2.

Example. Let the immigration process be stationary, i.e. $\{U_k, k \geq 1\}$ have a common distribution and $a = EU_k$ is finite. Then, due to weak law of large numbers, condition (18) is satisfied with $\xi = aEN$. Thus the Laplace transform of ξ is $e^{-\lambda aEN}$. From equality

$$\lambda aEN = \int_0^\infty \frac{1 - e^{-\lambda x}}{x} dP(x)$$

we obtain that measure $P(x)$ has only one atom of mass aEN at $x = 0$. Therefore $Q(x) = P * R(x) = a(1 - e^{-2x/B})$. From here denoting $\psi(\lambda) = -\log Ee^{-\lambda X}$ we have

$$\psi(\lambda) = \frac{2aEN}{B} \int_0^\infty \frac{1 - e^{-\lambda x EN}}{x} e^{-2x/B} dx,$$

consequently

$$\frac{d}{d\lambda}\psi(\lambda) = \frac{aENEW}{1 + BEW\lambda/2}.$$

By integration we obtain from the last equation that $\psi(\lambda) = \frac{2aEN}{B}\log(1 + \lambda BEW/2)$. We can see that in this case the limit distribution in Theorem 3 is gamma.

Corollary. *If $m = 1, B \in (0, \infty)$ and immigration is stationary with $a = EU_k < \infty$, then X_n/n as $n \to \infty$ has a gamma limit distribution with density function*

$$\frac{1}{\Gamma(\frac{2aE(N)}{B})} \left(\frac{2}{E(W)B}\right)^{\frac{2aE(N)}{B}} x^{\frac{2aE(N)}{B} - 1} e^{-\frac{2x}{E(W)B}}.$$

5. THE PROBABILITY OF NON EXTINCTION

In the case of stationary immigration $P\{X_n \neq 0\}$ approaches 1 as $n \to \infty$. However, if the immigration rate depends on the environment, this probability may approach to any number between 0 and 1 inclusively. Moreover, the asymptotic behavior of the process strongly depends on the behavior of this probability. Here we provide some results for $P\{X_n \neq 0\}$ in the case when the immigration rate approaches zero as $n \to \infty$.

Let $\gamma(n) = EU_n < \infty$ for each $n \geq 1$, regularly varies when $n \to \infty$ and $EW, EN, \alpha(n)$ and $\beta(n)$ are finite for each $n \geq 1$. From now on we also assume that

$$P\{U_n > 0\} = O(\gamma(n)), n \to \infty.$$

Theorem 4. *Let $m = 1, B \in (0, \infty)$ and $\gamma(n) \to 0, n \to \infty$. Then*
a) If $\gamma(n)\log n \to \infty$, then $P\{X_n \neq 0\} \to 1$;

b) If $\gamma(n)\log n \to 0, \beta(n) \to 0$, then $P\{X_n \neq 0\} \to 0$;
c) If $\gamma(n)\log n \to C \in (0,\infty)$, then $P\{X_n \neq 0\} \to 1 - \exp(-2CEN/B)$.

It is clear that when $\gamma(n)$ approaches zero "faster" than $(\log n)^{-1}$, the probability of non extinction may tend to zero arbitrarily. Next theorem gives the asymptotic behavior of that probability, which essentially determine the form of limit distribution of the process. We introduce two functions which are important in further considerations. Let

$$Q_1(n) = \frac{2EN}{B}\gamma(n)\log n, \quad Q_2(n) = \frac{2EN}{Bn}\sum_{k=1}^{n}\gamma(k).$$

Theorem 5. *If $m = 1, B \in (0,\infty), \gamma(n)\log n \to 0$ and $\beta(n) = o(Q_1(n) + Q_2(n))$, then as $n \to \infty$*

$$P\{X_n \neq 0\} \sim Q_1(n) + Q_2(n).$$

Examples. We consider some examples of possible asymptotic behavior of $P\{X_n \neq 0\}$. Let $\gamma(n) = C_1/n^\theta$.
a) If $\theta < 1$, then $\sum_{k=1}^{n}\gamma(k) \sim const\, n^{1-\theta}$ and $P\{X_n \neq 0\} \sim Q_1(n)$.
b) If $\theta > 1$, then $\sum_{k=1}^{n}\gamma(k) < \infty$ and $P\{X_n \neq 0\} \sim Q_2(n)$.
c) If $\theta = 1$, then $Q_1(n) \sim Q_2(n)$ and $P\{X_n \neq 0\} \sim 2Q_1(n)$.

Proof of Theorem 4. If we let $\lambda \to \infty$ in relation (6), we have

$$P\{X_n = 0\} = P\{U_n = 0\})\Psi(n, P_0), \tag{20}$$

where $P_0 = P\{W_{in} = 0\}, \Psi(n,s) = Es^{Z_n}, 0 \leq s \leq 1$. Since $\gamma(n) \to 0$, and $P\{U_n > 0\} = O(\gamma(n)), n \to \infty$, when $P_0 = 0$ we trivially obtain from (20) that $P\{X_n = 0\} \sim P\{Z_n = 0\}$. Assume that $0 < P_0 < 1$. We use the following probability generating functions of ξ_n and η_n instead of Laplace transforms introduced in Section 3:

$$g(s) = G(f(-\log s)), h_n(s) = H_n(f(-\log s))$$

for $0 \leq s \leq 1$. It is well known that $\Psi(n,s)$ can be represented as

$$\Psi(n,s) = \prod_{k=0}^{n} h_k(g_{n-k}(s)), \tag{21}$$

where $g_n(s)$ is nth functional iteration of $g(s)$ (see [3], for example). It is clear that $g_n(s)$ is the generating function of the BGW process without immigration and, when $m = 1, B \in (0, \infty)$,

$$1 - g_n(s) \sim \frac{1}{\frac{1}{1-s} + \frac{Bn}{2}} \tag{22}$$

as $n \to \infty$ for each $0 < s < 1$ (see [13], page 74). From here we conclude that $1 - g_n(P_0) \sim 1 - g_n(0), n \to \infty$. Consequently, taking this fact into account in (21), we obtain that $P\{Z_n = 0\} \sim \Psi(n, P_0)$ as $n \to \infty$ for each $0 \leq P_0 < 1$. Now the assertion of Theorem 4 follows from Lemma 3.1.1 in [13] (page 110), where asymptotic behavior of Z_n is studied in more general situation.

Proof of Theorem 5. We obtain from equation (20) the following relation

$$P\{X_n \neq 0\} = 1 - \Psi(n, P_0) + P\{U_n > 0\}\Psi(n, P_0). \tag{23}$$

The same arguments as in the proof of previous theorem give that $1 - \Psi(n, P_0) \sim P\{Z_n \neq 0\}$ as $n \to \infty$. It follows from Theorem 3.1.1 in mentioned above monograph [13] (page 108) that, when conditions of Theorem 5 are fulfilled,

$$P\{Z_n \neq 0\} \sim Q_1(n) + Q_2(n). \tag{24}$$

Now we consider the second summand on the right side of (23). Taking into account assumption $P\{U_n > 0\} = O(\gamma(n))$ we see that it is sufficient to show as $n \to \infty$

$$\gamma(n) = o(Q_1(n) + Q_2(n)). \tag{25}$$

Let first $Q_2(n) = o(Q_1(n))$. In this case clearly obtain that $\gamma(n) = o(Q_1(n))$. If $Q_1(n) = o(Q_2(n))$, then $\gamma(n) \log n = o(Q_2(n))$ and consequently $\gamma(n) = o(Q_2(n))$. When $Q_1(n) \sim const\ Q_2(n)$ we have $Q_1(n)+Q_2(n) \sim const\ \gamma(n) \log n$ and we can see again that relation (25) holds. Thus the assertion of the theorem follows from relations (23), (24) and (25).

Theorems 4 and 5 will be used in next section, where various limit distributions for process X_n will be derived. However these results are of independent interest as well. In particular Theorem 5 shows that event $\{X_n \neq 0\}$ may occur, roughly speaking, either because of descendants of "recent immigrants" or because of the individuals immigrated in the beginning of the process. For explanation of this phenomenon we refer to [13].

6. LIMIT DISTRIBUTIONS

In this section we obtain limit distributions for process X_n, when the immigration mean approaches to zero from generation to generation. We denote

$$a = \frac{2EN}{B}, \nabla(n) = \frac{2\alpha(n)}{B}.$$

Theorem 6. *If* $m = 1, B \in (0, \infty), \beta(n) \to 0$ *and* $\gamma(n) \to 0$ *such that* $\gamma(n) \log n \to \infty$*, then*

$$\lim_{n\to\infty} P\{(\frac{X_n}{n})^{\gamma(n)} \le x\} = x^a, 0 \le x \le 1.$$

If $\gamma(n) \log n \to C$, it follows from Theorem 4 that process X_n may extinct with positive probability. Therefore in this case we consider conditional process X_n, given $X_n > 0$.

Theorem 7.*If* $m = 1, B \in (0, \infty)$ *and* $\gamma(n) \log n \to C \in (0, \infty)$*, then*

$$\lim_{n\to\infty} P\{(X_n)^{\gamma(n)} \le x | X_n > 0\} = \frac{x^a - 1}{e^{aC} - 1}, 1 \le x \le e^C.$$

When $\gamma(n) \log n \to 0$, the form of the limit distribution depends on the behavior of function $\theta(n) = Q_1(n)/Q_2(n)$.

Theorem 8. *If* $m = 1, B \in (0, \infty), \gamma(n) \log n \to 0, \beta(n) = o(Q_1(n))$ *and* $\theta(n) \to \infty$*, then*

$$\lim_{n\to\infty} P\{\frac{\log X_n}{\log n} \le x | X_n > 0\} = x, 0 \le x \le 1.$$

Theorem 9. *If* $m = 1, B \in (0, \infty), \gamma(n) \log n \to 0, \beta(n) = o(Q_1(n))$ *and* $\theta(n) \to 0$*, then*

$$\lim_{n\to\infty} P\{\frac{2X_n}{Bn} \le x | X_n > 0\} = 1 - e^{-x}, x \ge 0.$$

When $\theta(n)$ has a positive finite limit we obtain two essentially different limit distributions having atoms.

Theorem 10. *If* $m = 1, B \in (0,\infty), \gamma(n)\log n \to 0, \beta(n) = o(Q_1(n))$ *and* $\theta(n) \to \theta \in (0,\infty)$*, then*

$$a) \lim_{n\to\infty} P\{\frac{\log X_n}{\log n} \le x | X_n > 0\} = \frac{x\theta}{1+\theta}, 0 \le x \le 1.$$

$$b) \lim_{n\to\infty} P\{\frac{2X_n}{Bn} \le x | X_n > 0\} = \frac{\theta + 1 - e^{-x}}{1+\theta}, x \ge 0.$$

It is not difficult to see that limit distribution in part $a)$ of last theorem has an atom of the mass $(1-\theta)^{-1}$ at point $x = 1$ and limit distribution in part $b)$ has an atom of the mass $\theta(1+\theta)^{-1}$ at point $x = 0$.

Proof of Theorem 6. We use the following result proved in [3] for the BGW processes.

Theorem B. *If* $m = 1, B \in (0,\infty), \alpha(n), \beta(n) \to 0$ *and* $\alpha(n)\log n \to \infty$*, then as* $n \to \infty$

$$P\{(\frac{Z_n}{n})^{\nabla(n)} \le x\} \to x,$$

where $0 \le x \le 1$.

Since $\nabla(n) \to 0$, we obtain from Theorem B that for any fixed $0 < y < \infty$ as $n \to \infty$

$$P\{(\frac{Z_n}{yn})^{\nabla(n)} \le x\} \to x.$$

This can be written as following

$$\frac{Z_n}{nx^{1/\nabla(n)}} \xrightarrow{\mathrm{D}} \xi$$

as $n \to \infty$ with $P\{\xi = 0\} = x = 1 - P\{\xi = \infty\}$ and consequently

$$E\exp\{-\frac{\lambda Z_n}{nx^{1/\nabla(n)}}\} \to Ee^{-\lambda\xi} = x. \tag{26}$$

Now we appeal to Theorem 2. Relation (26) shows that condition (16) is satisfied with $k(n) = nx^{1/\nabla(n)}$. If we choose $a(n) = k(n)$, then as $n \to \infty$

$$k(n)(1 - G(\frac{\lambda}{a(n)})) \to \lambda EW.$$

Thus condition (2) is fulfilled $b(\lambda) = \lambda EW$. Therefore due to Theorem 2 as $n \to \infty$

$$Ee^{-\lambda X_n/k(n)} \to Ee^{-\xi\lambda EW} = x. \tag{27}$$

We get the assertion of Theorem 6, if we write relation (27) in terms of the cumulative distribution function. Theorem 6 is proved.

Proof of Theorem 7. We use the following result from [3].

Theorem C. *If $m = 1, B \in (0, \infty)$, and $\alpha(n)\log n \to C \in (0, \infty)$, then as $n \to \infty$*

$$P\{\frac{(Z_n)^{\nabla(n)} - 1}{e^{2C/B} - 1} \leq x | Z_n > 0\} \to x,$$

where $0 \leq x \leq 1$.

Theorem C gives that, if $\alpha(n)\log n \to C$, then $\{Z_n/k(n)|Z_n > 0\} \to \xi$ in distribution as $n \to \infty$, where ξ has the same distribution that in proof of previous theorem and

$$k(n) = [x(e^{aC} - 1) + 1]^{1/\nabla(n)}.$$

Note here that $k(n) \to \infty$ when $n \to \infty$. Therefore

$$E[e^{-\lambda Z_n/k(n)}|Z_n > 0] \to Ee^{-\lambda\xi} = x$$

and condition (4) of Theorem 1 is satisfied. If we choose again $a(n) = k(n)$, we can easily see that condition (2) is also fulfilled with $b(\lambda) = \lambda EW$.

We need to show that $\Delta(n) \to 1$ as $n \to \infty$. We obtain from Theorem 4 that $P\{X_n > 0\} \to 1 - e^{-aC}$ when $\gamma(n)\log n \to C, n \to \infty$. On the other hand Lemma 3.1.1 in [15, page 110] gives that $P\{Z_n > 0\} \to 1 - e^{-2C/B}$ when $\alpha(n)\log n \to C, n \to \infty$. From these two results we conclude that $\Delta(n) \to 1$ as $n \to \infty$. Since $a(n) \to \infty$ and $P\{Z_n > 0\}$ has a positive limit, we easily see that $\delta(n, \lambda/a(n)) \to 0$ as $n \to \infty$ for each $\lambda > 0$. Hence all conditions of

Theorem 1 are satisfied and consequently $\{X_n/k(n)|X_n > 0\} \to \xi$ in distribution as $n \to \infty$. From here we obtain the assertion of Theorem 7.

Proofs of theorems 8-10 are similar to the proof of Theorem 7. Namely we show that conditions of Theorem 1 are fulfilled. This allows us to get the assertions of those theorems from limit theorems for the BGW process.

CONCLUDING REMARKS

Results obtained in this paper allow us to make the following conclusions. The asymptotic behavior of the process with continuous state space is similar to that of simple processes. Limit distributions for the new process can be obtained from corresponding limit theorems for BGW processes. In the case of conditional limit theorems one needs to check that the non-extinction probability for these two models have the same asymptotic behavior. The later is usually true when some quite natural assumptions are satisfied. The proofs of limit theorems consist of verifying conditions of the duality theorems proved in Section 2 of the paper.

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