Branching Processes with Decreasing Immigration and Tribal Emigration

I. Rahimov* and Walid S. Al-Sabah†

Abstract

At event times of a Poisson process a tribe of individuals enters a region where they sojourn for a random time according to a continuous time Markov branching process, and then emigrate from the region collectively. Conditions for the limit of the probability that the region contains at least one individual to be zero, one and a positive number less than one are provided in the critical case when the distribution of the size of immigrating tribe depends on the time of immigration. Limit distributions for the size of population in the region are also obtained.

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1 Introduction

We consider a family of independent, identically distributed and continuous time Markov branching processes. In a region $\Re$ let there be individuals of a single type. Each individual existing at a given moment, independently

*Institute of Mathematics of Uzbek Academy of Sciences, 700143, Tashkent and Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Box. 1339, Dhahran, 31261, Saudi Arabia. e-mail: rahimov @kfupm.edu.sa

†Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia. e-mail: walid@kfupm.edu.sa

of the fate of the other individuals, is transformed into k individuals with probability $\delta_{k1} + p_k \Delta t + o(\Delta t)$ in time $\Delta t \to 0$. Here $\delta_{k1} = 0, k \neq 1$ and $\delta_{11} = 1$. We assume that

$$p_1 < 0, p_k \geq 0, k \neq 1, \sum_{k \geq 0} p_k = 0$$

and denote by $Z(t)$ the number of individuals at time t given $Z(0) = 1$. We suppose A_i founders (immigrants) arrive in $\Re$ at a random time $T_i, i = 1, 2, \ldots$ Each of these founders initiates a family, that is, generates a continuous time Markov branching process with the same infinitesimal probabilities $\{p_k, k \geq 0\}$. The A_i founders arriving at time T_i collectively initiate a tribe which sojourns in $\Re$ for a time σ_i and then emigrates.

If we enumerate simultaneously arriving founders by $1, 2, ...$, then the pair (i, j) corresponds to the j-th founder arriving at time T_i. We shall call the branching process initiated by founder (i, j) as "(i, j)-process". Let $\{Z_{ij}(t), i, j \geq 1\}$ be the family of all possible (i, j)-processes. We assume that these processes are independent and identically distributed such that $Z_{ij}(t) \stackrel{d}{=} Z(t)$ for any $i, j \geq 1$. This means that processes $Z_{ij}(t)$ have the same infinitesimal probabilities $p_k, k \geq 0$. We define the process $W(t)$ by the relation

$$W(t) = \sum_{i=1}^{N(t)} \sum_{j=1}^{A_i} Z_{ij}(t - T_i) I_{\{}\sigma_i > t - T_{i\}}, \tag{1}$$

where $N(t) = \#\{i : T_i \leq t\}$, $I_{\{A\}}$ is the indicator of the event A. We assume that the sojourn times σ_i are i.i.d. non-negative random variables and are independent of the immigration and family components, $A_i, i = 1, 2, \ldots$, are independent random variables taking values $0, 1, \ldots$ and are also independent of the other components of the process. The $W(t)$ is the size of the population in $\Re$ at time t.

Note that since the indicator in (1) does not depend on j, the emigration of the tribe will occur collectively. In the case of family emigration, that is, when each family generated by a founder emigrates independently of the others, the process can be defined by a similar formula. In this case each (i, j)-process describing the evolution of a family has its sojourn time depending on i and j.

Immigration-birth-death-emigration processes with exponentially distributed sojourn times, $N(t)$ a Poisson process and $A_i \equiv 1$, were considered by Tessera (1984). Later, A. Pakes (1986) introduced a more general model where a random number of founders enters $\Re$ at the i-th event time T_i of a renewal process and generates i.i.d. Bellman-Harris processes. Considering the case of stationary immigration (i.e. $A_i, i \geq 1$, are i.i.d.), A. Pakes has obtained some sufficient or sufficient and necessary conditions of existence of a limiting distribution for the process $W(t)$ and has proved several limit theorems for the normalized process. The model introduced by Pakes is a contribution to the work on population processes with mass emigration or catastrophes some of which is reviewed by Brockwell (1985). An example of a general branching process with reproduction-dependent immigration and family emigration is considered by Rahimov (1992).

In this paper we shall consider the case of non-stationary immigration, that is, the random number of founders $A_i, i \geq 1$, which enters $\Re$ at the i-th event time T_i of a Poisson process are not identically distributed. More exactly, assuming that $EA_i \to 0$ as $i \to \infty$, we shall study the asymptotic behavior of the probability $P\{W(t) > 0\}$ as $t \to \infty$ and shall prove some new limit theorems for the normalized size of the population in $\Re$.

2 The probability of non-extinction

First we consider the "non-extinction" probability of the process $W(t)$. Let $R(t, S)$ be the probability generating function of the process $W(t)$. By the same arguments as Pakes (1986), we obtain under our assumptions the following relation:

$$R(t, S) = \exp\{-\frac{1}{\lambda_0}\int_0^t (1 - h(u, F(t-u, S)))\tau(t-u)du\}, \tag{2}$$

where

$$\tau(t) = 1 - P\{\sigma_i \leq t\}, F(t, S) = E[S^{Z_{ij}(t)} \mid Z_{ij}(0) = 1],$$

$$\lambda_0 = E[T_{i+1} - T_i], h(t, S) = E[S^{A_i} \mid T_i \in [t, t + \Delta t)], \Delta t \to 0.$$

It is known, Sevastyanov (1971), that the generating function $F(t, S)$ of the continuous time Markov branching process satisfies the following partial differential equation:

$$\frac{\partial F(t,S)}{\partial t} = f(F(t,S)), F(0,S) = S. \tag{3}$$

Here

$$f(S) = \sum_{k=0}^{\infty} p_k S^k, f(1) = 0$$

is the generating function of the infinitesimal probabilities. Assume further that $a = f'(1)$ and $b = f^{(2)}(1)$ are finite and

$$\alpha_0 = \sup_t \alpha(t) < \infty, \beta_0 = \sup_t \beta(t) < \infty, \tag{4}$$

where

$$\alpha(t) = \frac{\partial h(t,S)}{\partial S} |_{S=1}, \beta(t) = \frac{\partial^2 h(t,S)}{\partial S^2} |_{S=1} .$$

In addition we assume that $\alpha(t)$ is a regularly varying function, that is, it can be written as

$$\alpha(t) = \frac{l(t)}{t^{\alpha}}, \alpha \geq 0, t > 0, \tag{5}$$

and $l(t)$ varies slowly as $t \to \infty$. Let us define the function $b(t)$ by the relation

$$b(t) = \int_0^t (1 - F(u,0))\tau(u) du.$$

Since $t(1 - F(t,0))$ is bounded, under the condition $b \in (0, \infty)$ we have

$$\frac{b'(t)t}{b(t)} \to 0, t \to \infty. \tag{6}$$

It follows from (6) that $b(t)$ is a slowly varying function as $t \to \infty$ (Seneta (1976)). Now we shall prove the following theorem.

Theorem 1 *Let $a = 0$, $b \in (0, \infty)$ and $\alpha(t) \to 0$ as $t \to \infty$.*
a) If $\alpha(t)b(t) \to \infty, t \to \infty$, then

$$\lim_{t\to\infty} P\{W(t) > 0\} = 1;$$

b) if $\alpha(t)b(t) \to C$ and $\beta(t) \to 0$, $t \to \infty$, then

$$\lim_{t\to\infty} P\{W(t) > 0\} = 1 - \exp\{-\frac{C}{\lambda_0}\};$$

c) if $\alpha(t)b(t) \to 0$, $\beta(t) \to 0$, $t \to \infty$, *then*

$$\lim_{t\to\infty} P\{W(t) > 0\} = 0.$$

Examples. If the distribution of the sojourn times is exponential, then it is clear that $\alpha(t)b(t) \to 0$. If the distribution has a "heavy tail", for example, $\tau(t) \sim 1/\ln t$, then $b(t) \sim \frac{2}{b}\ln\ln t$. In this case the limit of the non-extinction probability may be either 1 or a positive number less than 1 depending on the asymptotic behavior of $\alpha(t)$.

Proof. Under condition (4) for any fixed $u \in [0, \infty)$, we have the following expansion for the generating function $h(u, S)$:

$$1 - h(u, S) = \alpha(t)(1 - S) - \frac{1}{2}\bar{\beta}(u, S)(1 - S)^2, \tag{7}$$

where $\mid \bar{\beta}(u, S) \mid \leq \beta(u)$, $\mid S \mid \leq 1$. Let us consider the case a). Since $b(t)$ slowly varies and $\alpha(t)b(t) \to \infty$, therefore $\alpha(t)$is also a slowly varying function.

Further, as is known, a slowly varying function $\alpha(x)$ admits the following representation

$$\alpha(x) = C_1(x)\exp\{\int_d^x \frac{\varepsilon(u)}{u} du\}, \tag{8}$$

where $d > 0$, $C_1(x) \to C_1 > 0$ and $\varepsilon(x) \to 0$as $x \to \infty$ (see Seneta (1976)).

From representation (8) it follows that for any slowly varying function $\alpha(t)$, there is a function $\lambda_\alpha(t) \to \infty$, $t \to \infty$, such that

$$\lim_{t\to\infty} \frac{\alpha(t/\lambda(t))}{\alpha(t)} = 1 \tag{9}$$

for any function $\lambda(t) \to \infty$, $1 \leq \lambda(t) \leq \lambda_\alpha(t)$. Using formulas (2) and (7) we obtain

$$\ln R(t, 0) = I_1 + I_2, \tag{10}$$

where

$$I_1 = -\frac{1}{\lambda_0}\int_0^t \alpha(u)(1 - F(t - u, 0))\tau(t - u)du,$$

$$I_2 = \frac{1}{2\lambda_0}\int_0^t \beta(u, F)(1 - F(t - u, 0))^2\tau(t - u)du \tag{11}$$

If we consider the estimate

$$I_1 \leq -\frac{1}{\lambda_0} \int_{t/\lambda_\alpha(t)}^{t} \alpha(u)(1 - F(t-u,0))\tau(t-u)du,$$

the integral on the right-hand side is non-greater than const $\alpha(t)b(t\theta_t)$, $\theta_t = 1 - (\lambda_\alpha(t))^{-1}$. Since $b(t)$varies slowly, we have that $I_1 \to -\infty$as $t \to \infty$.

The second summand in (10) is bounded by

$$\limsup_{t\to\infty} \lambda_0^{-1} \int_0^t \beta(t-u)(1 - F(u,0))^2 \tau(u)du < \infty.$$

Therefore $P\{W(t) = 0\} = R(t,0) \to 0$ as $t \to \infty$. Part a) is proved.

Let us prove part b. In this case we consider

$$-\lambda_0 \ln R(t,0) = R_1 + R_2, \tag{12}$$

where

$$R_1 = \int_0^{t/\lambda_\alpha(t)} (1 - h(u, F(t-u,0))\tau(t-u)du,$$

$$R_2 = \int_{t/\lambda_\alpha(t)}^{t} (1 - h(u, F(t-u,0))\tau(t-u)du.$$

Since, for any $u \in [0,\infty)$ and $\mid S \mid \leq 1$,

$$\mid 1 - h(u,S) \mid \leq \alpha(u) \mid 1 - S \mid, \tag{13}$$

we have

$$R_1 \leq \int_0^{t/\lambda_\alpha(t)} \alpha(u)(1 - F(t-u,0)\tau(t-u)du = o(1), t \to \infty. \tag{14}$$

Taking into account the choice of $\lambda_\alpha(t)$, we can see that as $t \to \infty$

$$\int_{t/\lambda_\alpha(t)}^{t} \alpha(u)(1 - F(t-u,0))\tau(t-u)du \sim \alpha(t)b(t\theta_t) \sim C \tag{15}$$

with $\theta_t = 1 - 1/\lambda_\alpha(t)$. On the other hand, if $\beta(t) \to 0$ as $t \to \infty$,

$$\int_{t/\lambda_\alpha(t)}^{t} \beta(u)(1 - F(t-u,0))^2 \tau(t-u)du \to 0, t \to \infty. \tag{16}$$

It follows from the relations (12), (14)-(16) and the formula (7) that

$$\lim_{t\to\infty} \lambda_0 \ln R(t,0) = -C. \tag{17}$$

Part b is proved.

By similar arguments it can be shown that if $\alpha(t)b(t) \to 0$, then the limit in (17) equals zero. Therefore $P\{W(t) = 0\} \to 1$as $t \to \infty$. The theorem is proved.

3 Limit theorems

First we consider the case $\alpha(t)b(t) \to \infty$, $t \to \infty$. If $b \in (0, \infty)$, it is known (Sevastyanov, 1971) that

$$P\{Z_{ij}(t) > 0\} = 1 - F(t,0) \sim \frac{2}{bt}. \tag{18}$$

We define the function $T(x)$ as follows:

$$T(x) = \exp\{\int_{t_0}^{x} \frac{\tau(u)}{u} du\}, x \geq 0, t_0 > 0.$$

It is clear that $T(x)$ is a slowly varying function, since it has the same form as the Karamata representation of slowly varying functions. It follows from (18) and the definition of $b(t)$ that

$$b(t) \sim \frac{2}{b} \ln T(t), t \to \infty. \tag{19}$$

It will be seen from our further considerations that the process may have distinct limiting distributions and normalizing functions depending on the rate of convergence of $\alpha(t)$ to zero. Therefore we introduce the following classes of processes which are defined by the limiting distribution $\pi(y)$ and the normalizing function $\varphi_t(x)$.

Definition We say that the process $X(t)$ belongs to the class $A(\varphi_t(x), \pi(y))$, if as $t \to \infty$

$$\varphi_t(X(t)) \xrightarrow{d} X, \quad P\{X \leq y\} = \pi(y).$$

We obtain the following result in the case when $\alpha(t)$ tends to zero as a slowly varying function.

Theorem 2 *If* $a = 0, b \in (0, \infty), \alpha(t) \to 0, \alpha(t)b(t) \to \infty$, *then*

$$W(t) \in A(\varphi_t(x), \quad E(y)),$$

where

$$\varphi_t(x) = \alpha(t) \ln \frac{T(t)}{T(x)}$$

and $E(y)$ *is the exponential distribution of the parameter* $2\lambda_0/b$.

Example. Let the distribution of the sojourn times be such that $\tau(t) \sim \Delta/\ln t, \Delta > 0$. It is not difficult to see that in this case

$$T(x) = [\ln x/\ln t_0]^{\Delta}, \varphi_t(x) = \Delta\alpha(t) \ln \frac{\ln t}{\ln x}$$

and we obtain the following corollary from Theorem 1.

Corollary 1. If conditions of Theorem 1 are satisfied and $\alpha(t) \ln\ln t \to \infty$, then for $0 \leq x \leq 1, \Delta > 0$

$$\lim_{t\to\infty} P\{(\frac{\ln W(t)}{\ln t})^{\alpha(t)} \leq x\} = x^{\Delta}.$$

Proof of Theorem 2. It follows from the condition $\alpha(t)b(t) \to \infty$ and (19) that $\varphi(x)_t \to \infty$ as $t \to \infty$ for any fixed x. For any fixed $0 < Z < \infty$ we choose a positive function C_t such that $C_t \to \infty$, $C_t < t$, for any $t \in (0, \infty)$ and $\varphi_t(C_t) \to Z$, as $t \to \infty$.

First we consider the integral

$$I = \int_0^t \alpha(u)(1 - F(t-u, S))\tau(t-u)du, \tag{20}$$

where $S = S_t = \exp\{-\theta/C_t\}$, $\theta \geq 0$. Let $\lambda_\alpha(t)$ be the function which satisfies (9). Integral (20) can be represented in the form

$$I \quad = \quad I_1 + I_2 + I_3, \tag{21}$$

where

$$I_1 = \int_0^{t/\lambda_\alpha} \alpha(u)(1 - F(t-u, S)\tau(t-u)du,$$

$$I_2 = \int_0^{C_t} \alpha(t-u)(1 - F(u, S))\tau(u)du,$$

$$I_3 = \int_{C_t}^{t-t/\lambda_\alpha} \alpha(t-u)(1-F(u,S))\tau(u)du.$$

Since $1-F(t,0)$ is a monotone function, we have that I_1 is non-greater than

$$\tau(t\theta_t)(1-F(t\theta_t,0)) \int_0^{t/\lambda_\alpha(t)} \alpha(u)du \to 0, \quad t \to \infty,$$

where $\theta_t = 1 - 1/\lambda_\alpha(t)$. On the other hand, using the inequality $1 - F(t,S) \leq 1 - S$, we obtain the following estimate for the second integral

$$I_2 \leq \alpha_0(1 - \exp\{-\frac{\theta}{C_t}\}) \int_0^{C_t} \tau(u)du \to 0 \ \ t \to \infty.$$

It remains to estimate I_3. It is known (see Sevastyanov, 1971) that if $a = 0, \ \ b \in (0,\infty)$, the generating function $F(t,S)$ can be represented in the following form:

$$1 - F(t,S) \quad = \quad \frac{1-S}{1+\frac{bt}{2}(1-S)}(1+\varepsilon(t,S)), \tag{22}$$

where $\varepsilon(t,S) \to 0$ uniformly on $\mid S \mid < 1$ as $t \to \infty$. Since $\alpha(t/\lambda(t)) \sim \alpha(t)$ as $t \to \infty$ for any function $1 \leq \lambda(t) \leq \lambda_\alpha(t)$, using (22),the third integral as $t \to \infty$ can be written in the form

$$I_3 = (1+o(1))\frac{2}{b}\alpha(t)[\ln\frac{T(\theta_t t)}{T(C_t)} \quad - \quad \int_{C_t}^{t-t/\lambda_\alpha(t)} \frac{\tau(u)}{u(1+\frac{bu}{2}(1-s))}du].$$

Taking into account the definition of the functions $\varphi_t(x)$ and $T(x)$ we obtain that

$$\alpha(t)\ln\frac{T(\theta_t t)}{T(C_t)} \quad = \quad \varphi_t(C_t) - \alpha(t)\int_{t-t/\lambda_\alpha(t)}^{t} \frac{\tau(u)}{u}du.$$

On the other hand

$$\int_{C_t}^{t-t/\lambda_\alpha(t)} \frac{\tau(u)}{u(1+\frac{bu}{2}(1-S))}du \leq \text{const} \ \ \tau(C_t).$$

From these arguments we conclude that

$$\lim_{t\to\infty} I_3 = \frac{2Z}{b}. \tag{23}$$

For any $\varepsilon > 0$ there exists $M > 0$ such that $\sup_{u>M} \tau(u) \leq \varepsilon/2\beta_0$. Further, for any fixed M and sufficiently large t

$$\int_0^M (1 - F(u, S_t))^2 du < \frac{\varepsilon}{2\beta_0}.$$

It follows from these facts that

$$\int_0^t \beta(u)(1 - F(t-u, S_t))\tau(u)du < \varepsilon$$

for sufficiently large t. Consequently, the integral on the left-hand side tends to zero as $t \to \infty$. Hence from relations (2), (7) and (23) we have

$$E \exp\{-\frac{W(t)\theta}{C_t}\} \quad \to \quad \exp\{-\frac{2Z}{b\lambda_0}\}. \tag{24}$$

According to the continuity theorem for Laplace transforms, (24) implies

$$P\{W(t) \leq xC_t\} \quad \to \quad \exp\{-\frac{2Z}{b\lambda_0}\} \tag{25}$$

for any $0 < x < \infty$. Putting $x = 1$ in (25), we have

$$\exp\{-\frac{2Z}{b\lambda_0}\} \quad = \quad \lim_{t\to\infty} P\{W(t) \leq C_t\} =$$

$$= \lim_{t\to\infty} P\{\varphi_t(W(t)) \geq \varphi_t(C_t)\} = \lim_{t\to\infty} P\{\varphi_t(W(t)) \geq Z\}.$$

The theorem is proved.

Now we consider the case $\alpha(t)b(t) \to C \in (0, \infty)$. We assume that $\tau(t) = P\{\sigma_i > t\}$ is a slowly varying function. Let

$$\varphi_t(x) = \frac{(T(x))^{\alpha(t)} - 1}{(T(t))^{\alpha(t)} - 1}, \quad x \geq 0,$$

$$\pi(\gamma, y) = \begin{cases} 0 & \text{if } y < 0 \\ [e^{-\gamma} + y(1 - e^{-\gamma})]^{\frac{2}{b\lambda_0}}, & \text{if } 0 \leq y < 1 \\ 1 & \text{if } y \geq 1 \end{cases}$$

Theorem 3 *If* $a = 0$, $b \in (0, \infty)$, $\alpha(t) \to 0$ *and* $\alpha(t)b(t) \to C \in (0, \infty)$ *as* $t \to \infty$, *then*

$$W(t) \in A(\varphi_t(x), \quad \pi(\frac{bC}{2}, y))$$

Note that the limiting distribution $\pi(\frac{bC}{2}, y)$ has an atom of the mass $\exp\{-C/\lambda_0\}$ at the point zero. It is clear that the appearance of the atom is caused by the fact that under the conditions of Theorem 3 the process has a positive "extinction" probability (see Theorem 1). By the arguments of the proof of this theorem one may find a limiting distribution that has no atom for the conditional distribution of the process and for the same normalizing function $\varphi_t(x)$.

Proof. Let $0 < \theta \leq 1$. We denote

$$k(t) = T^{-1}((1 + \theta(T^{\alpha(t)}(t) - 1))^{\frac{1}{\alpha(t)}}).$$

It is not difficult to see that for any $0 < \theta \leq 1$ function $k(t) \to \infty$ as $t \to \infty$. We consider

$$I = \frac{2}{b} \int_0^t \alpha(u) \frac{\tau(t-u)}{a(S) + t - u} du \tag{26}$$

where $a(S) = 2/b(1 - S)$, $S = \exp\{-Z/k(t)\}$, $Z \geq 0$.

Let $\lambda_\alpha(t)$ be the function which satisfies (9) and $\varepsilon > 0$. We represent (26) in the following form:

$$\frac{b}{2} I = I_1 + I_2 + I_3, \tag{27}$$

where

$$I_1 = \int_0^{t/\lambda_\alpha(t)} \alpha(u) \frac{\tau(t-u)}{a(S) + t - u} du,$$

$$I_2 = \int_{a(S)}^{r_1(t)} \alpha(t + a(S) - u)\tau(u - a(S))u^{-1} du,$$

$$I_3 = \int_{r_1(t)}^{r_2(t)} \alpha(t + a(S) - u)\tau(u - a(S))u^{-1} du,$$

$$r_1(t) = (1 + \varepsilon)a(S), \quad r_2(t) = t + a(S) - t/\lambda_{\alpha(t)}.$$

It is clear that I_1 is not greater than $t^{-1} \int_0^{t/\lambda_\alpha(t)} \alpha(u) du$ and thus tends to zero as $t \to \infty$. On the other hand, since

$$I_2 \leq \alpha_0 \log \frac{r_1(t)}{a(S)} = \alpha_0 \log(1 + \varepsilon),$$

the second summand is also arbitrarily small for sufficiently small ε.

In order to estimate I_3, we note that

$$\sup_{r_1(t)\le u\le r_2(t)} |\frac{\tau(u-a(S))}{\tau(u)}-1| \le \sup_{r_1(t)\le u\le r_2(t)} \sup_{\frac{\varepsilon}{1+\varepsilon}\le\Delta\le 1} |\frac{\tau(u\Delta)}{\tau(u)}-1|$$

and the last supremum tends to zero as $t\to\infty$ according to the uniform convergence theorem for slowly varying functions (see Seneta, 1976). Therefore as $t\to\infty$

$$I_3=(1+o(1))\alpha(t)\int_{r_1(t)}^{r_2(t)}\tau(u)u^{-1}du. \tag{28}$$

Since $T(t)$ is a slowly varying function and $a(S)\sim 2k(t)/bZ$, $t\to\infty$, $Z>0$, using the uniform convergence theorem, we have as $t\to\infty$

$$T(r_1(t))\sim T(a(S))\sim T(k(t)),\ \ T(r_2(t))\sim T(t).$$

Consequently it follows from (28) that

$$I_3=(1+o(1))\alpha(t)\ln\frac{T(t)}{T(k(t))}. \tag{29}$$

Hence from (27) and (29) we obtain

$$\lim_{t\to\infty} I=-\frac{2}{b}\ln[\exp\{-\frac{bC}{2}\}+\theta(1-\exp\{-\frac{bC}{2}\})] \tag{30}$$

As in the proof of Theorem 2 the integral

$$\int_0^t\beta(u)(1-F(t-u,S))^2\tau(t-u)du$$

tends to zero as $t\to\infty$ under condition (4) and $S=\exp\{-Z/k(t)\}$.

Using formulas (2), (7) and (22) and relation (30) we obtain that

$$\lim_{t\to\infty} E\exp\{-\frac{ZW(t)}{k(t)}\}=\pi(\frac{bC}{2},\ \ \theta),\ \ 0<\theta\le 1, z>0.$$

Hence we obtain

$$\pi(\frac{bC}{2},\ \ \theta)=\lim_{t\to\infty}P\{W(t)\le k(t)\}=\lim_{t\to\infty}P\{\varphi_t(W(t))\le\theta\}.$$

The theorem is proved.

We conclude our discussion with some comments on the asymptotic behavior of the expected number of individuals. It is not difficult to find from (2) by differentiating that

$$EW(t) = \frac{1}{\lambda_0} \int_0^t \alpha(u)\tau(t-u)du. \tag{31}$$

The last formula shows that the expected number of individuals strictly depends on the distribution of sojourn times. For example it tends to zero, if the sojourn times are exponentially distributed. If the immigration does not depend on the time, then it may tend to infinity or to a constant depending on the behavior of $\tau(x)$. Let both $\alpha(t)$ and $\tau(t)$ be regularly varying functions i.e.

$$\alpha(t) = \frac{l(t)}{t^\alpha}, \alpha \geq 0, \tau(t) = \frac{L(t)}{t^\delta}, \delta \geq 0, t > 0. \tag{32}$$

It is clear that $EW(t) \to 0$ if $\max(\alpha, \delta) > 1$. In the case $\alpha, \delta < 1$, it may have a different asymptotic behavior depending on $\alpha, \delta, l(t)$ and $L(t)$.

The following theorem gives the exact asymptotic behavior of the expectation in the case $\alpha, \delta \geq 0$.

Theorem 4 *Let $a = 0$ and $\alpha, \delta \geq 0$.*

a) If $0 < \alpha, \delta < 1$, then

$$EW(t) \sim \lambda_0^{-1} B(1-\alpha, 1-\delta)\alpha(t)\tau(t)t;$$

b) if at least one of α and δ is greater than or equal to 1, then

$$EW(t) \sim \lambda_0^{-1} \left[\tau(t)a_1(t) + \alpha(t)a_2(t)\right],$$

where $B(\alpha, \delta)$ is the β-function and

$$a_1(t) = \int_0^t \alpha(u)du, \quad a_2(t) = \int_0^t \tau(u)du.$$

Remark. It follows from Theorem 4 that if $\alpha + \delta > 1$, then the limit of the expectation is zero and if $\alpha + \delta < 1$, it tends to infinity. In the case $\alpha + \delta = 1$, the behavior of $EW(t)$ depends on the slowly varying functions $l(t)$ and $L(t)$.

Examples. Let the slowly varying functions $l(t)$ and $L(t)$ as $t \to \infty$ have finite limits C_1 and C_2 respectively.

1. In this case if $\alpha = \delta = \frac{1}{2}$, then we have

$$\lim_{t\to\infty} EW(t) = \frac{C_1 C_2}{\lambda_0}\, B\left(\frac{1}{2}, \frac{1}{2}\right).$$

2. If $\alpha = \delta = 1$, then

$$EW(t) \sim \frac{2C_1 C_2}{\lambda_0} \cdot \frac{\ln t}{t}.$$

3. If $\alpha = \delta = \frac{1}{3}$, then

$$EW(t) \sim \frac{C_1 C_2}{\lambda_0}\, B\left(\frac{2}{3}, \frac{2}{3}\right)\, \sqrt[3]{t}.$$

Proof of Theorem 4. First we prove part a). Let $r_1(t) = t/\lambda_l(t),\ r_2(t) = t/\lambda_L(t)$, where λ_l and λ_L are functions satisfying (9) for $l(t)$ and $L(t)$ respectively. If we write the integral in (31) as sum of integrals over $[0, r_1]$, $[r_1, t-r_2]$ and $[t-r_2, t]$, then the second integral is equivalent as $t \to \infty$ to

$$l(t)L(t)\int_{r_1}^{t-r_2} \frac{du}{u^\alpha (t-u)^\delta} \sim \alpha(t)\tau(t)t\, B(1-\alpha, 1-\delta).$$

Now we consider the integral over $[0, r_1]$. Let $\epsilon > 0$ such that $\alpha + \epsilon < 1$. It follows from the property of slowly varying functions (Seneta (1976), p. 20) that as $r_1 \to \infty$

$$\sup_{0\le u\le r_1} \{u^\epsilon l(u)\} \sim r_1^\epsilon l(r_1). \tag{33}$$

Thus for sufficiently large t

$$\int_0^{r_1(t)} \alpha(u)\tau(t-u)du \le \text{const }\ \alpha(t)\tau(t)t\left[\frac{1}{\lambda_\alpha(t)}\right]^{1-\alpha}.$$

By similar arguments we obtain that the integral over the third interval has also order $o(\alpha(t)\tau(t)t)$ as $t \to \infty$. Part a) of the theorem is proved.

Let us prove part b). Let $0 < \alpha < 1$ and $\delta \ge 1$. In this case $a_2(x)$ slowly varies as $x \to \infty$. If $\delta > 1$ it follows from the convergence of the integral. Let $\delta = 1$. Since $a_2'(x)x = L(x)$ and

$$\frac{L(x)}{a_2(x)} \le \frac{L(x)}{\displaystyle\int_{x/\lambda_L(x)}^{x} \tau(u)du} \sim \left(\ln \lambda_L(x)\right)^{-1}, \tag{34}$$

we have the relation $a_2'(x)x = o(a_2(x))$ that shows that $a_2(x)$ slowly varies in the case $\delta = 1$ also. Therefore there is a function $r(t) \to \infty,\ r(t) = o(t)$ such that $a_2(r(t)) \sim a_2(t),\ t \to \infty$. Now we consider

$$\int_0^t \alpha(u)\tau(t-u)du = I_1 + I_2 + I_3 \tag{35}$$

where $I_i,\ i = 1, 2, 3$ are integrals over the intervals $[0, t/\lambda_l(t)],\ [t/\lambda_l(t),\, t - r(t)]$ and $[t - r(t), t]$, respectively. It is not difficult to see that

$$I_3 \sim \alpha(t)a_2(r(t)) \sim \alpha(t)a_2(t). \tag{36}$$

Consider I_1. Using the form of $\tau(t)$ and the property (33), we obtain that for some constant $C > 0$

$$I_2 \leq C\tau(t)t\alpha(t)\left[\frac{1}{\lambda_l(t)}\right]^{1-\alpha}.$$

In the case $\delta = 1$ we obtain from (34) that $\tau(t)t = o(a_2(t))$. If $\delta > 1$, then $\tau(t)t = o(1)$, that is, we have again $\tau(t)t = o(a_2(t))$. Thus

$$I_2 = o\left(\alpha(t)a_2(t)\right), \quad t \to \infty. \tag{37}$$

It follows from the form of $\alpha(t)$ and $\tau(t)$ that there is a constant $C(\alpha, \delta) > 0$ such that

$$\inf_{\frac{t}{\lambda_l(t)} \leq u \leq t - r(t)} \left[\frac{\alpha(t)}{\alpha(u)} + \frac{\tau(t)}{\tau(t-u)}\right] \geq C(\alpha, \delta). \tag{38}$$

Using this fact we have

$$I_2 \leq \frac{1}{C(\alpha, \delta)}\left[A_1 + A_2\right]$$

where

$$\begin{aligned} A_1 &= \alpha(t)\int_{r(t)}^{t-\lambda_l(t)} \tau(u)du = o\left(\alpha(t)a_2(t)\right) \\ A_2(t) &\sim \alpha(t)\tau(t)t = o\left(\alpha(t)a_2(t)\right). \end{aligned}$$

Thus we have in the case $0 < \alpha < 1$ and $\delta \geq 1$ the expectation $EW(t) \sim \lambda_0^{-1}\alpha(t)a_2(t)$. If $0 < \delta < 1,\ \alpha \geq 1$, then repeating the above arguments we obtain that $EW(t) \sim \lambda_0^{-1}\tau(t)a_1(t)$. Let now $\alpha = \delta = 1$. In this case both $a_1(t)$ and $a_2(t)$ are slowly varying functions as $t \to \infty$. Therefore there are functions $r_i(t) \to \infty,\ r_i(t) = o(t), \quad i = 1, 2$, such that

$$a_i\left(r_i(t)\right) \sim a_i(t), \ \ t \to \infty, \quad i = 1, 2.$$

If we partition the interval $[0, t]$ as

$$[0, t] = E_1 \cup E_2 \cup E_3$$

where $E_1 = [0, r_1]$, $E_2 = [r_1, t - r_2]$, and $E_3 = [t - r_2, t]$, then the integrals over E_1 and E_3 are equivalent to $\tau(t)a_1(t)$ and $\alpha(t)a_2(t)$ respectively. Using (38) again, we obtain that the integral over E_2 has the order $o(\alpha(t)a_2(t) + \tau(t)a_1(t))$. The theorem is proved.

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