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Pre-variational Inequalities in Banach Spaces[1]

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Abstract: In this paper, we study the existence of solution of pre-variational inequality in the setting of reflexive Banach spaces. We see that the existence theorems of Yang and Chen (Ref.6) are also hold good in the setting of reflexive Banach spaces with some less assumptions.

Key Words: Variational inequality problem, pre-variational inequality problem, KKM-maps, reflexive Banach spaces.

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1. Introduction

Let E be a normed space with its dual E^* and $K \subset E$ be a closed convex subset. We denote the pairing between E^* and E by $\langle \cdot, \cdot \rangle$. Given two maps $T : K \longrightarrow E^*$ and $\eta : K \times K \longrightarrow E$, then the *pre-variational inequality problem* is to find $u \in K$ such that

$$\langle T(u),\ \eta(v, u) \rangle \ \geq\ 0, \qquad \forall v \in K. \tag{1.1}$$

We note that problem (1.1) was investigated in Refs.(4,6), where E is the n-dimensional Euclidean space.

Let $\psi : K \longrightarrow I\!R$ be Fréchet differentiable. Then ψ is called $\eta - convex$ on K (Ref.3) if there exists a continuous map $\eta : K \times K \longrightarrow E$ such that

$$\psi(v) - \psi(u)\ \geq\ \langle \psi'(u),\ \eta(v, u) \rangle, \qquad \forall u, v \in K,$$

where $\psi'(u)$ is the Fréchet derivative of ψ at u.

Suppose that f is $\eta - convex$ on K for some continuous map $\eta : K \times K \longrightarrow E$. Then the minimization problem

$$\min\ f(u) \qquad \text{subject to} \quad u \in K, \tag{1.2}$$

where f is Fréchet differentiable with $f'(u) = T(u)$, has a solution u if it is a solution of pre-variational inequality problem (1.1) (Refs.1,4).

In this paper, we prove the existence of solution of (1.1) in the setting of reflexive Banach spaces. We see that the existence Theorems 9 and 10 in Ref.6 are also hold good in the setting of reflexive Banach spaces without the assumption that $\eta(.,.)$ is normal and regular, respectively, on K in these theorems.

2. Existence Results

Through out in this section, we will consider X as a real reflexive Banach space with its dual X^*. The bilinear form $\langle \cdot, \cdot \rangle$ is supposed to be continuous.

We need the following concepts and results.

A multivalued map $S : X \longrightarrow 2^X$ is called $KKM - map$, if for every finite subset $\{u_1, u_2, \cdots, u_n\}$ of X, $conv(\{u_1, u_2, \cdots, u_n\}) \subset \bigcup_{i=1}^n S(u_i)$,
where 2^X is the set of all subsets of X and $conv(A), \ \ \forall A \subset X$, the convex hull of A.

Lemma 2.1 (Ref.2). Let A be an arbitrary nonempty set in a topological vector space Y and $S : A \longrightarrow 2^Y$ be a KKM-map. If $S(u)$ is closed for all $u \in A$ and is compact for at least one $u \in A$ then

$$\bigcap_{u \in A} S(u) \neq \emptyset.$$

We now establish the main result of this paper.

Theorem 2.1. Let K be a nonempty closed convex bounded subset of a real reflexive Banach space X. Assume that:

(i) $\eta(u, u) = 0, \ \ \forall u \in K;$

(ii) $h(v) := \langle T(u), \ \eta(v, u)\rangle$ is convex in $v, \ \ \forall u \in K;$

(iii) $g(u) := \langle T(u), \ \eta(v, u)\rangle$ is upper semicontinuous and concave in $u, \ \ \forall v \in K.$

Then there exists an element $u_0 \in K$ such that

$$\langle T(u_0), \ \eta(v, u_0)\rangle \ \geq \ 0, \qquad \forall v \in K.$$

Proof. Let

$$F(v) = \{u \in K : \langle T(u), \ \eta(v, u)\rangle \ \geq \ 0\}, \qquad \forall v \in K.$$

We note that since $\eta(v, v) = 0, \ \ \forall v \in K$, we have $v \in F(v)$ and hence $F(v)$ is nonempty, $\ \forall v \in K$.

By the same arguments as in Theorem 2.1 of Ref.5, we see that F is a KKM-map. We remark that $F(v)$ is closed, $\ \ \forall v \in K$. Indeed, let $\{u_n\}$ be a sequence in $F(v)$ such that $u_n \longrightarrow u \in K$. Since $u_n \in F(v) \ \ \forall n$, we have

$$\langle T(u_n), \ \eta(v, u_n)\rangle \ \geq \ 0, \qquad \forall v \in K.$$

Since $g(u) = \langle T(u), \ \eta(v, u)\rangle$ is upper semicontinuous, $\ \forall v \in K$, we have

$$\limsup_{n \to \infty} \langle T(u_n), \ \eta(v, u_n)\rangle \leq \langle T(u), \ \eta(v, u)\rangle$$

and hence

$$\langle T(u),\ \eta(v,u)\rangle \geq 0, \qquad \forall v \in K.$$

This implies that $u \in F(v)$ and hence $F(v)$ is norm closed.
Next we will see that $F(v)$ is convex. Let $u_1, u_2 \in F(v)$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Then we have

$$\langle T(u_1),\ \eta(v,u_1)\rangle \ \geq\ 0 \tag{2.1}$$

and

$$\langle T(u_2),\ \eta(v,u_2)\rangle \ \geq\ 0. \tag{2.2}$$

Multiplying (2.1) by α and (2.2) by β and adding the resulting inequalities, we have

$$\alpha\langle T(u_1),\ \eta(v,u_1)\rangle + \beta\langle T(u_2),\ \eta(v,u_2)\rangle \ \geq\ 0, \qquad \forall v \in K.$$

Since $g(u) = \langle T(u),\ \eta(v,u)\rangle$ is concave, $\forall v \in K$, we have

$$\begin{aligned}\langle T(\alpha u_1 + \beta u_2),\ \eta(v, \alpha u_1 + \beta u_2)\rangle &= g(\alpha u_1 + \beta u_2)\\ &\geq \alpha g(u_1) + \beta g(u_2)\\ &= \alpha\langle T(u_1),\ \eta(v,u_1)\rangle + \beta\langle T(u_2),\ \eta(v,u_2)\rangle\\ &\geq 0.\end{aligned}$$

This implies that $\alpha u_1 + \beta u_2 \in F(v)$ and hence $F(v)$ is convex.

Now, we equip X with the weak topology. Then K, as a closed bounded convex subset in the real reflexive Banach space X, is weakly compact. Since $F(v)$ is a closed convex subset of a reflexive Banach space, $F(v)$ is weakly closed. From the facts that $F(v) \subset K$ and weak closedness of $F(v)$, we have $F(v)$ is weakly compact. Then by Lemma 2.1, we have

$$\bigcap_{v \in K} F(v) \neq \emptyset.$$

Hence there exists $u_0 \in K$ such that

$$\langle T(u_0),\ \eta(u,u_0)\rangle \ \geq\ 0, \quad \forall u \in K.$$

If we take $X = \mathbb{R}^n$, then there is no need to prove that $F(v)$ is convex, and hence we have the following result:

Corollary 2.1. Let K be closed bounded convex subset of $\mathbb{R}^n$ and $T : K \longrightarrow \mathbb{R}^n$. Assume that

(i) $\eta(u,u) = 0, \quad \forall u \in K$;

(ii) $h(v) := \langle T(u),\ \eta(v,u)\rangle$ is convex in $v, \quad \forall u \in K$;

(iii) $g(u) := \langle T(u),\ \eta(v,u)\rangle$ is upper semicontinuous in $u, \quad \forall v \in K$.

Then there exists an element $u_0 \in K$ such that

$$\langle T(u_0),\ \eta(v,u_0)\rangle \ \geq\ 0, \qquad \forall v \in K.$$

Remark 2.1. Corollary 2.1 covers Theorem 8 in Ref.6 as a special case where T is assumed to be continuous.

A map $T : K \longrightarrow X^*$ is said to be *pre-coercive with respect to* $\eta(y,x)$ (Ref.6) if there exists $u_0 \in K$ such that

$$\frac{\langle T(u) - T(u_0),\ \eta(u,u_0)\rangle}{||\eta(u,u_0)||} \longrightarrow +\infty, \tag{2.3}$$

whenever $||u|| \longrightarrow +\infty$.

When K is not bounded, we have the following result:

Theorem 2.2. Let K be a nonempty closed convex subset of a real reflexive Banach space X. Assume that:

(i) $\eta(u,u) = 0, \quad \forall u \in K$;

(ii) $T : K \rightarrow X^*$ is pre-coercive with respect to $\eta(v,u)$;

(iii) $h(v) := \langle T(u),\ \eta(v,u)\rangle$ is convex in v for each fixed $u \in K$;

(iv) $g(u) := \langle T(u),\ \eta(v,u)\rangle$ is upper semicontinuous and concave in $u, \quad \forall v \in K$.

Then the pre-variational inequality (1.1) has solutions.

Proof: As in Ref.6, let B_r denote the closed ball of centre 0 and radius r in X. If we substitute $K \bigcap B_r$ for K then the assumptions of Theorem 2.1 are satisfied. Hence there exists a solution u_r of the following per-variational inequality problem:

$$\text{Find } \ u \in K \cap B_r \ \text{ such that } \ \langle T(u),\ \eta(v,u)\rangle \geq 0, \quad \forall v \in K \cap B_r.$$

Choose $r \geq ||u_0||$ with u_0 as in the pre-coercive condition. Then

$$\langle T(u_r),\ \eta(u_0,u_r)\rangle \geq 0 \tag{2.4}$$

As in Ref.6, we have

$$\langle T(u_r),\ \eta(u_0,u_r)\rangle \leq ||\eta(u_0,u_r)||[\frac{-\langle T(u_0)-T(u_r),\ \eta(u_0,u_r)\rangle}{||\eta(u_0,u_r)||} + ||T(u_0)] \tag{2.5}$$

If $||u_r|| = r$ for all r, we choose r large enough that (2.5) and condition (2.3) imply

$$\langle T(u_r),\ \eta(u_0,u_r) < 0$$

which is a contradiction of (2.4). Hence, there exists r such that $||u_r|| < r$. For each $v \in K$, we choose $\alpha > 0$ small enough that $\alpha v + (1-\alpha)u_r \in K \cap B_r$. Therefore

$$\langle T(u_r),\ \eta(\alpha v + (1-\alpha)u_r, u_r) \geq 0.$$

Since $h(v) = \langle T(u),\ \eta(v,u)\rangle$ is convex in v, $\forall u \in K$, we have

$$\alpha\langle T(u_r),\ \eta(v,u_r)\rangle + (1-\alpha)\langle T(u_r),\ \eta(u_r,u_r)\rangle \geq 0.$$

But by assumption (i), $\eta(u_r,u_r) = 0$, and hence

$$\langle T(u_r),\ \eta(v,u_r) \geq 0, \quad \forall v \in K.$$

Hence u_r is a solution of pre-variational inequality problem (1.1).

Now, again if $X = I\!R^n$ then from Corollary 2.1 and Theorem 2.2, we have the following result:

Corollary 2.2. Let K be a nonempty closed convex subset of $I\!R^n$. Assume that:

(*i*) $\eta(u,u) = 0, \quad \forall u \in K$;

(*ii*) $T : K \longrightarrow I\!R^*$ is pre-coercive with respect to $\eta(v,u)$;

(*iii*) $h(v) := \langle T(u),\ \eta(v,u)\rangle$ is convex in v for each fixed $u \in K$;

(*iv*) $g(u) := \langle T(u),\ \eta(v,u)\rangle$ is upper semicontinuous in $u, \quad \forall v \in K$.

Then the pre-variational inequality (1.1) has solutions.

Remark 2.2. We note that Corollary 2.2 generalizes Theorems 9 and 10 in Ref.6. In this corollary, we have neither assumed that $\eta(v,u)$ is normal nor regular on K as these conditions were taken in Ref.6.

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