Southeast Asian Bulletin of Mathematics (2004) 27: 973–982

Southeast Asian
Bulletin of
Mathematics

On Existence of Pareto Equilibria for Constrained Multiobjective Games*

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AMS Mathematics Subject Classification(2000): 90D10

Abstract. In this paper, we study the existence of weighted Nash equilibria and Pareto equilibria for the constrained multiobjective games with or without involving Φ -condensing map. Our results improve and unify the corresponding results of the multiobjective games in the literature.

Keywords: Constrained multiobjective game, weighted Nash equilibria, Pareto equilibria, fixed point

1. Introduction

In 1950, Nash [11] (see, also [12]) introduced the concept of equilibrium point in n-person games. Debreu [6] extended this concept of Nash equilibrium point for n-person games to constrained equilibrium problems. In the last three decades two problems, n-person games and n-person games with constrained were extensively studied in the literature; See, for example, [1, 2] and references therein. In the recent past, much attention has been paid on the game theory with vector payoff; See, for example, [3, 4, 9, 14, 15, 16, 18, 19, 20] and references therein. The existence of Pareto equilibria is one of the fundamental problem in game theory.

Wang [16] introduced the concept of a weighted Nash equilibrium of a multiobjective game and proved that any normalized weighted Nash equilibrium is a weak Pareto equilibrium of a multiobjective game. He also formulated the weighted Nash equilibrium in terms of fixed points of a multivalued map. Wang [16], Yu and Yuan [18], and Yuan and Tarafdar [19] used this formulation to prove the existence of weighted Nash equilibrium and Pareto equilibrium of a

*This research was partially done during the visit of the first author to the Abdus Salam ICTP, Trieste, Italy, and also supported by the Major Research Project of University Grants Commission, New Delhi, Govt. of India.

multiobjective game by using some fixed point and minimax theorems. Recently, Ding [8] studied the multiobjective games with constrained correspondences. He proved the existence of weighted Nash equilibrium and Pareto equilibrium for the constrained multiobjective games in the setting of H-spaces.

In this paper, we establish some existence results of weighted Nash equilibrium and Pareto equilibrium for the constrained multiobjective games with or without involving Φ-condensing maps. Our results improve and unify the corresponding results of the multiobjective games in the literature.

2. Preliminaries

Throughout the paper, we follow the terminology of Ding [8] and, Yu and Yuan [18]. For each given $m \in \mathbb{N}$, we denote by $\mathbb{R}^m_+$ the non-negative orthant of $\mathbb{R}^m$, that is,

$$\mathbb{R}^m_+ = \{u = (u_1, \ldots, u_m) \in \mathbb{R}^m : \ u_j \geq 0 \text{ for } j = 1, \ldots, m\},$$

so that $\mathbb{R}^m_+$ has a nonempty interior with the topology induced in terms of convergence of vectors with respect to the Euclidean metric. That is,

$$\text{int } \mathbb{R}^m_+ = \{u = (u_1, \ldots, u_m) \in \mathbb{R}^m : \ u_j > 0 \text{ for } j = 1, \ldots, m\}.$$

We denote by $\mathbb{T}^m_+$ and int $\mathbb{T}^m_+$ the simplex of $\mathbb{R}^m_+$ and its relative interior, respectively, that is,

$$\mathbb{T}^m_+ = \left\{ u = (u_1, \ldots, u_m) \in \mathbb{R}^m_+ : \sum_{j=1}^m u_j = 1 \right\},$$

$$\text{int } \mathbb{T}^m_+ = \left\{ u = (u_1, \ldots, u_m) \in \text{int } \mathbb{R}^m_+ : \sum_{j=1}^m u_j = 1 \right\}.$$

Let I be a finite index set, that is, $I = \{1, \ldots, n\}$ and for each $i \in I$, p_i a positive integer. For each $i \in I$, let X^i be a nonempty subset of a topological vector space E^i, $X = \prod_{i \in I} X^i$ and $X^{\hat{i}} = \prod_{j \in I, j \neq i} X^j$. For each $x \in X$, $x^i \in X^i$ denotes the ith coordinate, $x^{\hat{i}} = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) \in X^{\hat{i}}$ and we write $x = (x^{\hat{i}}, x^i)$.

We consider a constrained game with finite players and multicriteria in its strategic form $\Gamma := (X^i, A^i, F^i)_{i \in I}$. For each player $i \in I$, X^i is its strategy set; $A^i : X^{\hat{i}} \to 2^{X^i}$ is its constrained correspondence which restricts the strategies of the ith player to the subset $A^i(x^{\hat{i}}) \subset X^i$ when all the players have chosen their strategies $x^j \in X^j$, $j \neq i$, and $F^i = (f^i_1, f^i_2, \ldots, f^i_{p_i}) : X \to \mathbb{R}^{p_i}$ is its payoff function (or say, loss function or multicriteria). In such a constrained multiobjective game, the other players influence player $j \in I$

(a) indirectly, by restricting js feasible strategies to $A^j(x^{\hat{j}})$,
(b) direct, by affecting js payoff function F^j.

If a choice $x = (x^1, \dots, x^n) \in X$ is played, each player i is trying to minimize her/his payoff function $F^i(x) = (f_1^i(x), f_2^i(x), \dots, f_{p_i}^i(x))$, which consists of non-commensurable outcomes. Each player i has a preference $\succcurlyeq_i$ over the outcome space $\mathbb{R}^{p_i}$. For each player $i \in I$, its preference $\succcurlyeq_i$ is given as follows:

$$z^1 \succcurlyeq_i z^2 \quad \text{if and only if} \quad z_j^1 \geq z_j^2,$$

for each $j = 1, \dots, p_i$, where $z^1 = (z_1^1, \dots, z_{p_i}^1)$ and $z^2 = (z_1^2, \dots, z_{p_i}^2)$ are any elements of $\mathbb{R}^{p_i}$. The players' preference relations induce the preferences on X, defined for each player i and choose $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^n) \in X$ by $x \succcurlyeq_i y$, whenever $F^i(x) \succcurlyeq_i F^i(y)$. In the constrained multiobjective game, each player $i \in I$ is trying to minimize her/his own payoff according to her/his preferences.

If $A^i(x^{\hat{i}}) = X^i$ for each $i \in I$ and for all $x^{\hat{i}} \in X^{\hat{i}}$, then the model of constrained multiobjective games reduced to the model of multicriteria games $G = (X^i, F^i)_{i \in I}$ considered and studied by Wang [15, 16], Ding [7], Yuan and Tarafdar [19] and Yu and Yuan [18] and references therein. If for each player $i \in I$, $F^i(x) = f^i(x)$, that is, $p_i = 1$, which consists of commensurable outcomes, then the model of the constrained multiobjective games reduces to the model of the constrained games (or say, metagames); see, for example, [1, 2] and references therein.

For the games with vector payoff functions (or multicriteria), as it is well known, in general, there does not exist a strategy $\bar{x} \in X$ to minimize (or equivalent to say, maximize) all f_j^is for each player $i \in I$; See, for example [17]. Hence, we need to recall some solution concepts for the constrained multicriteria games.

Definition 2.1. [8] *A strategy $\bar{x}^i \in X^i$ of player i is called a Pareto efficient strategy (respectively, a weak Pareto efficient strategy) with respect to $\bar{x} \in X$ if, $\bar{x}^i \in A^i(\bar{x}^{\hat{i}})$ and there is no strategy $y^i \in A^i(\bar{x}^{\hat{i}})$ such that $F^i(\bar{x}^{\hat{i}}, \bar{x}^i) - F^i(\bar{x}^{\hat{i}}, y^i) \in \mathbb{R}_+^{p_i} \setminus \{\mathbf{0}\}$ (respectively, $F^i(\bar{x}) - F^i(\bar{x}^{\hat{i}}, y^i) \in \operatorname{int} \mathbb{R}_+^{p_i}$).*

Definition 2.2. [8] *A strategy combination $\bar{x} \in X$ is called a Pareto equilibrium (respectively, a weak Pareto equilibrium) of the constrained multiobjective game $\Gamma = (X^i, A^i, F^i)_{i \in I}$ if, for each player i, $\bar{x}^i \in A^i(\bar{x}^{\hat{i}})$ is a Pareto efficient strategy (respectively, a weak Pareto efficient strategy) with respect to $\bar{x}$.*

It is clear that each Pareto equilibrium is certainly a weak Pareto equilibrium, but the converse need not be true. We need the following concept which is introduced by Ding [8].

Definition 2.3. *A strategy combination $\bar{x} \in X$ is called a weighted Nash equilibrium with weight combination $W = (W^1, W^2, \dots, W^n)$ of a constrained multiobjective game $\Gamma = (X^i, A^i, F^i)_{i \in I}$ if, for each player $i \in I$, we have that*

(i) $\bar{x}^i \in A^i(\bar{x}^{\hat{i}})$;

(ii) $W^i \in \mathbb{R}_+^{p_i} \setminus \{\mathbf{0}\}$;

(iii) $W^i \cdot F^i(\bar{x}^{\hat{i}}, \bar{x}^i) \leq W^i \cdot F^i(\bar{x}^{\hat{i}}, y^i)$, *for all $y^i \in A^i(\bar{x}^{\hat{i}})$,* *where $\cdot$ denotes the inner product in $\mathbb{R}^{p_i}$. In particular, when $W^i \in \mathbb{T}_+^{p_i}$, for all $i \in I$, the strategy $\bar{x} \in X$ is called a normalized weighted Nash equilibrium with respect to W.*

From the above definition, it is easy to verify that a strategy $\bar{x} \in X$ is a weighted Nash equilibrium with respect to the weight vector $W = (W^1, W^2, \ldots, W^n)$ of the constrained multiobjective game $\Gamma = (X^i, A^i, F^i)$ if and only if $\bar{x} \in X$ is an optimal solution of the following constrained optimization problem:

(COP) Find $\bar{x} \in X$ such that for each $i \in I, \bar{x}^i \in A^i(\bar{x}^{\hat{i}})$ and $W^i \cdot F^i(\bar{x}^{\hat{i}}, \bar{x}^i) = \min_{y^i \in A^i(\bar{x}^{\hat{i}})} W^i \cdot F^i(\bar{x}^{\hat{i}}, y^i)$.

We need the following lemma of Ding [8] which tells us that the existence problems of Pareto equilibrium for constrained multiobjective games can be reduced to the existence of the weighted Nash equilibrium under certain circumstances.

Lemma 2.4. [8] *Each normalized weighted Nash equilibrium $\bar{x} \in X$ with respect to a weight vector $W = (W^1, W^2, \ldots, W^n) \in \mathbb{T}_+^{k_1} \times \cdots \times \mathbb{T}_+^{k_n}$ (respectively, $W = (W^1, W^2, \cdots, W^n) \in \text{int } \mathbb{T}_+^{k_1} \times \cdots \times \text{int } \mathbb{T}_+^{k_n}$) for a constrained multiobjective game $\Gamma = (X^i, A^i, F^i)$ is a weak Pareto equilibrium (respectively, a Pareto equilibrium) of the game Γ.*

Let B be nonempty subset of a topological vector space Z, then we denote by $co(B)$ the convex hull of B. A subset B of a topological space E is said to be *compactly open* (respectively, *compactly closed*) in E if, for any nonempty compact subset D of E, $B \cap D$ is open (respectively, closed) in D.

Definition 2.5. [13] *Let E be a Hausdorff topological vector space and L a lattice with least element, denoted by $\mathbf{0}$. A mapping $\Phi : 2^E \to L$ is called a measure of noncompactness provided that the following conditions hold for any $M, N \in 2^E$:*

(i) $\Phi(M) = \mathbf{0}$ *if and only if M is precompact, i.e. it is relatively compact.*

(ii) $\Phi(\overline{\text{co}}M) = \Phi(M)$*, where $\overline{\text{co}}M$ denotes the closed convex hull of M.*

(iii) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

It follows from (iii) that if $M \subseteq N$, then $\Phi(M) \leq \Phi(N)$.

Definition 2.6. [13] *Let $\Phi : 2^E \to L$ be a measure of noncompactness on E and $D \subseteq E$. A multivalued map $Q : D \to 2^E$ is called Φ-condensing provided that if $M \subseteq D$ with $\Phi(Q(M)) \geq \Phi(M)$ then M is relatively compact.*

Remark 2.7. Note that every multivalued map defined on a compact set is necessarily Φ-condensing. If E is locally convex, then a compact multivalued map (i.e., $Q(D)$ is precompact) is Φ-condensing for any measure of noncompactness Φ. Obviously, if $Q : D \to 2^E$ is Φ-condensing and if $Q' : D \to 2^E$ satisfies $Q'(x) \subseteq Q(x)$ for all $x \in D$, then Q' is also Φ-condensing.

We shall use the following particular form of a fixed point theorem due to Chowdhury and Tan [5].

Theorem 2.8. *Let K be a nonempty convex subset of a topological vector space (not necessarily, Hausdorff) E and $S : K \to 2^K$ a multivalued map. Assume that the following conditions hold:*

(i) *For all $x \in K$, $S(x)$ is nonempty and convex.*

(ii) *For all $y \in K$, $S^{-1}(y) = \{x \in K : y \in S(x)\}$ is comapctly open.*

(iii) *There exist a nonempty, closed and compact (not necessarily convex) subset D of K and a $\tilde{y} \in D$ such that $K \setminus D \subset S^{-1}(\tilde{y})$.*

Then there exists $\bar{x} \in K$ such that $\bar{x} \in S(\bar{x})$.

Remark 2.9. If K is a nonempty closed convex subset of a Hausdorff topological vector space X, then the condition (iii) of Theorem 2.8 can be replaced by the following condition (see, for example, [10, Corollary 2]).

(iii)′ The multivalued map $S : K \to 2^K$ is Φ-condensing.

3. Existence of Weighted Nash Equilibrium and Pareto Equilibrium

Rest of the paper, unless otherwise specified, we assume that $X = \prod_{i \in I} X^i$ with the product topology.

Theorem 3.1. *Let $\Gamma = (X^i, A^i, F^i)$ be a constrained multiobjective game, where for each player $i \in I$, X^i is a nonempty, closed and convex subset of a Hausdorff topological vector space E^i, $A^i : X^{\hat{i}} \to 2^{X^i}$ is the constrained correspondence, and $F^i = (f_1^i, f_2^i, \dots, f_{p_i}^i) : X \to \mathbb{R}^{p_i}$ is the payoff function. Assume that there exists a weight vector $W = (W^1, W^2, \dots, W^n)$ with $W^i \in \mathbb{R}^{p_i} \setminus \{\mathbf{0}\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) *For each $i \in I$, $A^i : X^{\hat{i}} \to 2^{X^i}$ is a multivalued map with nonempty and convex values and for each $y^i \in X^i$, $(A^i)^{-1}(y^i)$ is compactly open in X. Further, we assume that the set $\mathcal{D} = \{x \in X : x \in A(x)\}$ is compactly closed in X, where $A : X \to 2^X$ is Φ-condensing multivalued map defined as $A(x) = \prod_{i \in I} A^i(x^{\hat{i}})$ for all $x \in X$.*

(ii) *The function $(x, y) \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.*

(iii) *For each fixed $y \in X$, the mapping $x \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is upper semicontinuous on each compact subset of X.*

(iv) *For each fixed $x \in X$, the mapping $y \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is quasi-convex on X.*

Then the constrained multiobjective game Γ has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W^i \in \operatorname{int} \mathbb{T}_+^{p_i}$, then Γ has a Pareto equilibrium.

Proof. For the sake of simplicity, we define a bifunction $F : X \times X \to \mathbb{R}$ by

$$F(x, y) := \sum_{i \in I} W^i \cdot \left[F^i(x^{\hat{i}}, x^i) - F^i(x^{\hat{i}}, y^i)\right] \text{ for all } x, y \in X.$$

For each $x \in X$, define a multivalued map $P : X \to 2^X$ by

$$P(x) = \{y \in X : F(x,y) > 0\}.$$

Then by condition (iv), $P(x)$ is convex for all $x \in X$. From condition (ii) and (iii), for all $y \in X$, the complement of $P^{-1}(y)$ in X

$$P^{-1}(y) = \{x \in X : F(x,y) \leq 0\}$$

is compactly closed in X and therefore $[P^{-1}(y)]^c$ is compactly open in X.

Since for each $i \in I$ and for all $x \in X$, $A^i(x^{\hat{i}})$ is nonempty and convex, we have $A(x) = \prod_{i\in I} A^i(x^{\hat{i}})$ is nonempty and convex. Also since for all $y \in X$, $A^{-1}(y) = \bigcap_{i\in I}(A^i)^{-1}(y^i)$ and $(A^i)^{-1}(y^i)$ is compactly open for each $i \in I$ and for all $y^i \in X^i$, it follows that $A^{-1}(y)$ is compactly open in X for all $y \in X$.

Now assume that for all $x \in \mathcal{D}$, $A(x) \cap P(x) \neq \emptyset$. Define another multivalued map $S : X \to 2^X$ by

$$S(x) = \begin{cases} A(x) \cap P(x) & \text{if } x \in \mathcal{D}, \\ A(x) & \text{if } x \in K \setminus \mathcal{D}. \end{cases}$$

Then S has nonempty and convex values and by [8, Lemma 3.2], $S^{-1}(y)$ is compactly open in X for all $y \in X$. Since $S(x) \subseteq A(x)$, for all $x \in K$ and A is Φ-condensing, by Remark 2.7 we have, S is also Φ-condensing. Hence by Theorem 2.8 along with Remark 2.9, there exists $\hat{x} \in X$ such that $\hat{x} \in S(\hat{x})$. From the definition of $\mathcal{D}$ and S, we have $\{x \in X : x \in S(x)\} \subseteq \mathcal{D}$. Therefore, $\hat{x} \in \mathcal{D}$ and $\hat{x} \in A(\hat{x}) \cap P(\hat{x})$ and, in particular, we get

$$0 = F(\hat{x},\hat{x}) = \sum_{i\in I} W^i \cdot \left[F^i(\hat{x}^{\hat{i}},\hat{x}^i) - F^i(\hat{x}^{\hat{i}},\hat{x}^i)\right] > 0,$$

a contradiction. Hence there exists $\bar{x} \in \mathcal{D}$ such that $A(\bar{x}) \cap P(\bar{x}) = \emptyset$, that is, $\bar{x} \in A(\bar{x})$ and $F(\bar{x},y) = \sum_{i\in I} W^i \cdot \left[F^i(\bar{x}^{\hat{i}},\bar{x}^i) - F^i(\bar{x}^{\hat{i}},y^i)\right] \leq 0$ for all $y \in A(\bar{x})$. For each $i \in I$ and for any given $y^i \in A^i(\bar{x}^{\hat{i}})$, let $y = (\bar{x}^{\hat{i}}, y^i)$, then we have $y \in A(\bar{x})$ and it follows from last inequality that

$$W^i \cdot F^i(\bar{x}^{\hat{i}},\bar{x}^i) \leq W^i \cdot F^i(\bar{x}^{\hat{i}},y^i) \text{ for all } y^i \in A^i(\bar{x}^{\hat{i}}).$$

This proves that for each $i \in I$, $\bar{x}^i \in A^i(\bar{x}^{\hat{i}})$ and $W^i \cdot F^i(\bar{x}^{\hat{i}},\bar{x}^i) = \min_{y^i\in A^i(\bar{x}^{\hat{i}})} W^i \cdot F^i(\bar{x}^{\hat{i}},y^i)$, that is, $\bar{x} \in X$ is a weighted Nash equilibrium point for the constrained multiobjective game Γ with respect to weight vector W.

Lemma 2.4 shows that $\bar{x}$ is also a weak Pareto equilibrium of Γ, and Pareto equilibrium point of Γ if $W^i \in \operatorname{int} \mathbb{T}^{p_i}_+$ for all $i \in I$. ■

Corollary 3.2. *Let $\Gamma = (X^i, A^i, F^i)$ be a constrained multiobjective game, where for each player $i \in I$, X^i is a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space E^i, $A^i : X^{\hat{i}} \to 2^{X^i}$ is the constrained correspondence, and $F^i = (f^i_1, f^i_2, \ldots, f^i_{p_i}) : X \to \mathbb{R}^{p_i}$ is the payoff function. Assume that there exists a weight vector $W = (W^1, W^2, \ldots, W^n)$ with $W^i \in \mathbb{R}^{p_i} \setminus \{\mathbf{0}\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) *For each $i \in I$, $A^i : X^{\hat{i}} \to 2^{X^i}$ is a multivalued map with nonempty and convex values and for each $y^i \in X^i$, $(A^i)^{-1}(y^i)$ is compactly open in X. Further, we assume that the set $\mathcal{D} = \{x \in X : x \in A(x)\}$ is compactly closed in X, where $A : X \to 2^X$ is a compact multivalued map defined as $A(x) = \prod_{i \in I} A^i(x^{\hat{i}})$ for all $x \in X$.*

(ii) *The function $(x, y) \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.*

(iii) *For each fixed $y \in X$, the mapping $x \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is upper semicontinuous on each compact subset of X.*

(iv) *For each fixed $x \in X$, the mapping $y \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is quasi-convex on X.*

Then the constrained multiobjective game Γ has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W^i \in \operatorname{int} \mathbb{T}_+^{p_i}$, then Γ has a Pareto equilibrium.

Proof. Since for each $i \in I$, E^i is locally convex and $A : X \to 2^X$ is a compact multivalued map, by Remark 2.7 A is Φ-condensing map and the conclusion follows from Theorem 3.1. ∎

For each $i \in I$ and for all $x^{\hat{i}} \in X^{\hat{i}}$, if $A^i(x^{\hat{i}}) = X^i$ then we have the following existence results of weighted Nash equilibrium and Pareto equilibrium for multiobjective games.

Corollary 3.3. *Let $G = (X^i, F^i)$ be a multiobjective game, where for each player $i \in I$, X^i is a nonempty compact convex subset of a Hausdorff topological vector space E^i and $F^i = (f_1^i, f_2^i, \ldots, f_{p_i}^i) : X \to \mathbb{R}^{p_i}$ is the payoff function. Assume that there exists a weight vector $W = (W^1, W^2, \ldots, W^n)$ with $W^i \in \mathbb{R}^{p_i} \setminus \{\mathbf{0}\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) *The function $(x, y) \mapsto \sum_{i \in} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.*

(ii) *For each fixed $y \in X$, the mapping $x \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is upper semicontinuous on each compact subset of X.*

(iii) *For each fixed $x \in X$, the mapping $y \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is quasi-convex on X.*

Then the multiobjective game G has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W^i \in \operatorname{int} \mathbb{T}_+^{p_i}$, then G has a Pareto equilibrium.

Proof. For each $i \in I$ and for all $x^{\hat{i}} \in X^{\hat{i}}$, let $A^i(x^{\hat{i}}) = X^i$. Since for each $i \in I$, X^i is compact, the multivalued map $A : X \to 2^X$ defined as $A(x) = \prod_{i \in I} A^i(x^{\hat{i}})$ for all $x \in X$, is Φ-condensing. Then the conclusion follows from Theorem 3.1. ∎

Corollary 3.4. *Let $G = (X^i, F^i)_{i \in I}$ be a given multiobjective game. For each $i \in I$, let X^i be a nonempty and convex subset of a Hausdorff topological vector* *space E^i. If there is a weight vector $W = (W^1, W^2, \dots, W^n)$ with $W^i \in \mathbb{R}_+^{p_i}\setminus\{\mathbf{0}\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) *The function $(x, y) \mapsto \sum_{i\in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.*

(ii) *For each fixed $y \in X$, the mapping $x \mapsto \sum_{i\in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is upper semicontinuous on each compact subset of X.*

(iii) *For each fixed $x \in X$, $\sum_{i\in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is quasi-convex on X.*

(iv) *There exists a nonempty, compact and convex subset D^i of X^i such that for each $x \in X \setminus D$, there exists $\tilde{y}^i \in D^i$ such that $W^i \cdot \left[F^i(x^{\hat{i}}, x^i)) - F^i(x^{\hat{i}}, \tilde{y}^i)\right] > 0$, where $D = \prod_{i\in I} D^i \subseteq X$.*

Then the multiobjective game G has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W^i \in \operatorname{int} \mathbb{T}_+^{p_i}$, then the game G has a Pareto equilibrium.

Proof. For each $i \in I$, let $\{y_1^i, \dots, y_k^i\}$ be a finite subset of X^i. Let $C^i = co(D^i \cup \{y_1^i, \dots, y_k^i\})$. Then for each $i \in I$, C^i is nonempty, compact and convex. Then by Corollary 3.3, there exists $\bar{x} \in C = \prod_{i\in I} C^i$ such that for each $i \in I$,

$$W^i \cdot \left[F^i(\bar{x}^{\hat{i}}, \bar{x}^i) - F^i(\bar{x}^{\hat{i}}, y^i)\right] \le 0 \text{ for all } y^i \in C^i.$$

From condition (iv), $\bar{x} \in D$. In particular, $\bar{x} \in D$ such that for each $i \in I$,

$$W^i \cdot \left[F^i(\bar{x}^{\hat{i}}, \bar{x}^i) - F^i(\bar{x}^{\hat{i}}, y_k^i)\right] \le 0 \text{ for all } k.$$

For each $i \in I$ and for all $y^i \in X^i$, we now define

$$Q(y^i) = \{x \in D : W^i \cdot \left[F^i(x^{\hat{i}}, x^i) - F^i(x^{\hat{i}}, y^i)\right] \le 0\}.$$

From conditions (i) and (ii), $Q(y^i)$ is closed for all $y^i \in X^i$. Hence every finite subfamily of closed sets $Q(y_i)$ has nonempty intersection. Since D is compact, for each $i \in I$, $\bigcap_{y_i \in D^i} Q(y_i) \neq \emptyset$. And the result is proved. ■

Remark 3.5. For each $i \in I$, if E^i is a reflexive Banach space equipped with the weak topology, the assumption (iv) in Corollary 3.4 can be replaced by the following condition.

(iv)′ There exists an $r > 0$ such that for all $x \in X$, $||x|| \ge r$, there exists $\tilde{y}_i \in X^i$, $||\tilde{y}_i||_i < r$ such that $W^i \cdot \left[F^i(x^{\hat{i}}, x^i) - F^i(x^{\hat{i}}, \tilde{y}^i)\right] > 0$, where $||\cdot||_i$ and $||\cdot||$ denote the norms on $\mathbb{R}^{p_i}$ and $\prod_{i=1}^n \mathbb{R}^{p_i}$, respectively.

Proof. Define $B_i^r = \{x_i \in X^i : ||x_i||_i \le r\}$. Then B_i^r is a nonempty, compact and convex subset of X^i. By taking $D^i = B_i^r$ in Corollary 3.4, we get the conclusion. ■

For each $i \in I$, when X^i is not necessarily Hausdorff and the multivalued map $A : X \to 2^X$ defined by $A(x) = \prod_{i\in I} A^i(x^{\hat{i}})$ for all $x \in X$ is not necessarily Φ-condensing, we have the following results.

Theorem 3.6. *Let $\Gamma = (X^i, A^i, F^i)$ be a constrained multiobjective game, where for each player $i \in I$, X^i is a nonempty convex subset of a topological vector space (not necessarily, Hausdorff) E^i, $A^i : X^{\hat{i}} \to 2^{X^i}$ is the constrained correspondence, and $F^i = (f_1^i, f_2^i, \dots, f_{p_i}^i) : X \to \mathbb{R}^{p_i}$ is the payoff function. Assume that there exists a weight vector $W = (W^1, W^2, \dots, W^n)$ with $W^i \in \mathbb{R}^{p_i} \setminus \{\mathbf{0}\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) *For each $i \in I$, $A^i : X^{\hat{i}} \to 2^{X^i}$ is a multivalued map with nonempty and convex values and for each $y^i \in X^i$, $(A^i)^{-1}(y^i)$ is compactly open in X. Further, we assume that the set $\mathcal{D} = \{x \in X : x \in A(x)\}$ is compactly closed in X, where $A(x) = \prod_{i \in I} A^i(x^{\hat{i}})$ for all $x \in X$.*

(ii) *The function $(x, y) \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.*

(iii) *For each fixed $y \in X$, the mapping $x \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is upper semicontinuous on each compact subset of X.*

(iv) *For each fixed $x \in X$, the mapping $y \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is quasi-convex on X.*

(v) *For each $i \in I$, there exist a nonempty compact (not necessarily, convex) subset D^i of X^i and $\tilde{y}^i \in D^i$ such that for all $x \in X \setminus D$, $\tilde{y}^i \in A^i(x^{\hat{i}})$ and $W^i \cdot [F^i(x)) - F^i(x^{\hat{i}}, \tilde{y}^i)] > 0$, where $D = \prod_{i \in I} D^i \subseteq X$.*

Then the constrained multiobjective game Γ has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W^i \in \operatorname{int} \mathbb{T}_+^{p_i}$, then Γ has a Pareto equilibrium.

Proof. It is easy to see that the condition (v) implies condition (iii) of Theorem 2.8, and the result follows from the proof of Theorem 3.1 by using Theorem 2.8. ∎

Corollary 3.7. *Let $G = (X^i, F^i)$ be a multiobjective game, where for each player $i \in I$, X^i is a nonempty convex subset of a topological vector space (not necessarily, Hausdorff) E^i and $F^i = (f_1^i, f_2^i, \dots, f_{p_i}^i) : X \to \mathbb{R}^{p_i}$ is the payoff function. Assume that there exists a weight vector $W = (W^1, W^2, \dots, W^n)$ with $W^i \in \mathbb{R}^{p_i} \setminus \{\mathbf{0}\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) *The function $(x, y) \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.*

(ii) *For each fixed $y \in X$, the mapping $x \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is upper semicontinuous on each compact subset of X.*

(iii) *For each fixed $x \in X$, the mapping $y \mapsto \sum_{i \in I} W^i \cdot F^i(x^{\hat{i}}, y^i)$ is quasi-convex on X.*

(iv) *For each $i \in I$, there exist a nonempty compact (not necessarily, convex) subset D^i of X^i and $\tilde{y}^i \in D^i$ such that for all $x \in X \setminus D$, $W^i \cdot [F^i(x)) - F^i(x^{\hat{i}}, \tilde{y}^i)] > 0$, where $D = \prod_{i \in I} D^i \subseteq X$.*

Then the multiobjective game G has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W^i \in \operatorname{int} \mathbb{T}_+^{p_i}$, then G has a Pareto equilibrium.

Proof. For each $i \in I$ and for all $x^{\hat{i}} \in X^{\hat{i}}$, let $A^i(x^{\hat{i}}) \in X^i$, then we get the conclusion from Theorem 3.6. ∎

Remark 3.8. Corollaries 3.3, 3.4 and 3.7 generalize Theorems 3 and 4 in [18] in several ways.

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