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Fixed Point Theorems for Non-convex Valued Multifunctions

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Abstract. The purpose of this paper is to establish fixed point theorems for non-convex valued multifunctions which generalize known results in the literature. We also derive coincidence theorems in the non-compact setting.

Keywords: Fixed point theorems, coincidence theorems, Hausdorff topological vector spaces

1. Introduction and Preliminaries

We shall use the following notation and definitions. Let A be a non-empty set. We shall denote by 2^A the family of all subsets of A. If A is a non-empty subset of a topological vector space X, we shall denote by $int_X(A)$ and $co(A)$ the interior of A in X and the convex hull of A, respectively. Let X and Y be two topological vector spaces and let $F : X \to 2^Y$ be a multifunction. The *inverse* of F, denoted by F^{-1}, is the multifunction from $\mathscr{R}(F)$, the range of F, to X defined by

$$x \in F^{-1}(y) \quad \text{if and only if } y \in F(x).$$

In 1968, Browder [3] proved the following fixed point theorem.

Theorem A. [3, Theorem 2, p. 285] *Let K be a non-empty compact convex subset of a Hausdorff topological vector space X, and let $S : K \to 2^K$ be a multifunction such that*

(a) *for each $x \in K$, $S(x)$ is non-empty and convex,*
(b) *for each $y \in K$, $S^{-1}(y)$ is open in K.*

Then T has a fixed point, that is, there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

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Because of the applications in nonlinear analysis, variational inequalities, optimization, operations research and economics, Theorem A has been generalized in many different directions, see for example [1–10]. First in 1977, Tarafdar [7] improved Theorem A by weakening assumption (b). Then in 1987, he generalized his result for non-compact setting. The following fixed point theorem generalizes Theorem A, Theorem 1 in [7], Corollary 3 in [9] and Theorem 3.3 in [10].

Theorem B. [1, 6] *Let K be a non-empty compact convex subset of a Hausdorff topological vector space X, and let $S, T : K \to 2^K$ be two multifunctions. Assume that*

(a) *for each $x \in K$, $co(S(x)) \subseteq T(x)$ and $S(x)$ is non-empty,*
(b) $K = \bigcup\{int_K\, S^{-1}(y) : y \in K\}$.

Then T has a fixed point, that is, there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

The main motivation of this paper is to generalize a result of Tarafdar [8] for non-convex valued multifunctions and Theorem B for non-compact setting. We also derive coincidence theorems in the non-compact setting.

2. Fixed Point Theorems

In this section, we prove the following fixed point theorem which improves a result of Tarafdar [8] and Theorems A and B.

Theorem 2.1. *Let K be a non-empty convex subset of a Hausdorff topological vector space X, and let $S, T : K \to 2^K$ be two multifunctions. Assume that*

(a) *for each $x \in K$, $co(S(x)) \subseteq T(x)$ and $S(x)$ is non-empty,*
(b) $K = \bigcup\{int_K\, S^{-1}(y) : y \in K\}$,
(c) *there exists a non-empty subset B_0 of K such that B_0 is contained in a compact convex subset B_1 of K and the set $\mathfrak{D} = \bigcap\{K \backslash int_K\, S^{-1}(y) : y \in B_0\}$ is either empty or compact.*

Then there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

Proof. We first assume that $\mathfrak{D} = \emptyset$ and define a multifunction $G : B_1 \to 2^{B_1}$ as $G(x) = S(x) \cap B_1$, for all $x \in B_1$.

For each $x \in B_1$, $G(x)$ is non-empty. Indeed, suppose that $G(x)$ is empty for some $x \in B_1$. Then $S(\tilde{x}) \cap B_1 = \emptyset$ for some $\tilde{x} \in B_1$. Hence for all $\bar{x} \in B_1, \bar{x} \notin S(\tilde{x})$ and therefore $\tilde{x} \notin S^{-1}(\bar{x}) \supseteq int_K\, S^{-1}(\bar{x})$. This implies that $\tilde{x} \in K \backslash int_K\, S^{-1}(\bar{x})$ for all $\bar{x} \in B_1$ and hence $\tilde{x} \in \bigcap_{\bar{x} \in B_1}\{K \backslash int_K\, S^{-1}(\bar{x})\}$. Therefore $\mathfrak{D}$ is non-empty, a contradiction of our assumption. Moreover, we have

(a$_1$) For all $x \in B_1$, $co(G(x)) = co(S(x) \cap B_1) \subseteq (co(S(x)) \cap co(B_1)) \subseteq (T(x) \cap B_1) \subseteq T(x)$ and hence $co(G(x)) \subseteq T(x)$ for all $x \in B_1$.
(b$_1$) Since $\mathfrak{D} = \bigcap\{K \backslash int_K\, S^{-1}(y) : y \in B_0\} = \emptyset$, from assumption (b) we have, $K = \bigcup\{int_K\, S^{-1}(y) : y \in B_0\}$ and hence $K = \bigcup\{int_K\, S^{-1}(y) : y \in B_1\}$. By noting that for each $y \in B_1, G^{-1}(y) = S^{-1}(y) \cap B_1$ and $int_K\, S^{-1}(y) \cap B_1 \subseteq$

$int_{B_1}(S^{-1}(y) \cap B_1)$, we have

$$\bigcup_{y \in B_1} \{int_{B_1} G^{-1}(y)\} = \bigcup_{y \in B_1} \{int_{B_1}(S^{-1}(y) \cap B_1)\}$$
$$\supseteq \bigcup_{y \in B_1} \{int_K(S^{-1}(y) \cap B_1)\} = K \cap B_1 = B_1.$$

Therefore $\bigcup_{y \in B_1}\{int_{B_1} G^{-1}(y)\} = B_1$.

Thus from Theorem B, there exists $x_0 \in B_1$ such that $x_0 \in T(x_0)$.

Now we will consider the case when $\mathfrak{D}$ is a non-empty compact subset of K. Assume that T has no fixed point. We divide the remaining proof into four parts.

(1) *Claim:* $K \backslash int_K S^{-1}(y) \neq \varnothing$ *for all* $y \in K$.

Suppose that $K \backslash int_K S^{-1}(y) = \varnothing$ for some $y \in K$, then $y \notin K \backslash int_K S^{-1}(y)$. This implies that $y \in int_K S^{-1}(y) \subseteq S^{-1}(y)$ and thus $y \in S(y) \subseteq co(S(y)) \subseteq T(y)$. Therefore y is a fixed point of T, a contradiction of our assumption. Hence $K \backslash int_K S^{-1}(y) \neq \varnothing$ for all $y \in K$.

(2) *Claim: the convex hull of each finite subset* $\{y_1, y_2, \ldots, y_n\}$ of K *is contained in the union* $\bigcup_{i=1}^n \{K \backslash int_K S^{-1}(y_i)\}$.

Let $\{y_1, y_2, \ldots, y_n\}$ be a finite subset of K and $\alpha_i \geq 0$ for each $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \alpha_i = 1$. Suppose that

$$\hat{x} = \sum_{i=1}^n \alpha_i y_i \notin \bigcup_{i=1}^n \{K \backslash int_K S^{-1}(y_i)\}.$$

Then $\hat{x} \in int_K S^{-1}(y_i)$ for $i = 1, 2, \ldots, n$. Thus $\hat{x} \in S^{-1}(y_i)$ for each $i = 1, 2, \ldots, n$ and hence $y_i \in S(\hat{x}) \subseteq co(S(\hat{x}))$ for each $i = 1, 2, \ldots, n$. Therefore

$$\sum_{i=1}^n \alpha_i y_i = \hat{x} \in co(S(\hat{x})).$$

This implies that $\hat{x} \in co(S(\hat{x})) \subseteq T(\hat{x})$ and thus $\hat{x}$ is a fixed point of T, which contradicts to our assumption. Hence the convex hull of each finite subset $\{y_1, y_2, \ldots, y_n\}$ of K is contained in the union $\bigcup_{i=1}^n \{K \backslash int_K S^{-1}(y_i)\}$.

(3) *Claim:* $\bigcap_{y \in A}\{K \backslash int_K S^{-1}(y)\} \neq \varnothing$, where $A = co(B_1 \cup \{y_1, y_2, \ldots, y_n\})$ and $\{y_1, y_2, \ldots, y_n\}$ is a finite subset of K.

Since $A = co(B_1 \cup \{y_1, y_2, \ldots, y_n\})$, A is compact and convex. Suppose that $\bigcap_{y \in A}\{K \backslash int_K S^{-1}(y)\} = \varnothing$. Then we can define a multifunction $Q : A \to 2^A$ by

$$Q(x) = \{y \in A : x \notin K \backslash int_K S^{-1}(y)\}$$

such that $Q(x)$ is non-empty, for each $x \in A$. For $y \in A$,

$$Q^{-1}(y) = \{x \in A : y \in Q(x)\} = \{x \in A : x \notin K \backslash int_K S^{-1}(y)\}$$
$$= \{x \in A : x \in int_K S^{-1}(y)\} = int_K S^{-1}(y) \cap A.$$

We now define another multifunction $P : A \to 2^A$ by

$$P(x) = co\big(Q(x)\big), \quad \text{for all } x \in A.$$

We will show that $A = \bigcup_{y \in A}\{int_A\, Q^{-1}(y)\}$. Since $\bigcap_{y \in A}\{K \backslash int_K\, S^{-1}(y)\} = \emptyset$, we have $\bigcup_{y \in A}\{int_K\, S^{-1}(y)\} = K$. Hence

$$A \supseteq \bigcup_{y \in A}\{int_A\, Q^{-1}(y)\} \supseteq \bigcup_{y \in A}\{int_K\, S^{-1}(y) \cap A\} = K \cap A = A.$$

By Theorem B, there exists $x_0 \in A$ such that $x_0 \in P(x_0) = co\big(Q(x_0)\big)$.

This implies that there exists a finite subset $\{y_1, y_2, \ldots, y_k\}$ of A such that $y_i \in Q(x_0)$ for $i = 1, 2, \ldots, k$, where $x_0 = \sum_{i=1}^k \alpha_i y_i$, $\alpha_i \geq 0$ for $i = 1, 2, \ldots, k$ and $\sum_{i=1}^k \alpha_i = 1$. This means that $x_0 \notin K \backslash int_K\, S^{-1}(y_i)$ for all $i = 1, 2, \ldots, k$, that is, $x_0 \in int_K\, S^{-1}(y_i)$ for all $i = 1, 2, \ldots, k$ and hence $x_0 = \sum_{i=1}^k \alpha_i y_i \in \bigcap_{i=1}^k \{int_K\, S^{-1}(y_i)\}$, which contradicts to our Claim (2). Thus $\bigcap_{y \in A}\{K \backslash int_K\, S^{-1}(y)\} \neq \emptyset$.

(4): From Claim (3), we have

$$\begin{aligned}
\mathfrak{D} \cap \left(\bigcap_{i=1}^{n}\{K \backslash int_K\, S^{-1}(y_i)\}\right) &= \left(\bigcap_{y \in B_0}\{K \backslash int_K\, S^{-1}(y)\}\right) \cap \left(\bigcap_{i=1}^{n}\{K \backslash int_K\, S^{-1}(y_i)\}\right) \\
&\supseteq \bigcap_{y \in A}\{K \backslash int_K\, S^{-1}(y)\}, \text{ as } B_0 \cup \{y_1, y_2, \ldots, y_n\} \subseteq A \\
&\neq \emptyset.
\end{aligned}$$

That is, for each finite subset $\{y_1, y_2, \ldots, y_n\}$ of K, $\mathfrak{D} \cap \left(\bigcap_{i=1}^{n}\{K \backslash int_K\, S^{-1}(y_i)\}\right) \neq \emptyset$. Since $\mathfrak{D}$ is compact and $\{K \backslash int_K\, S^{-1}(y)\}$ is closed, $\{K \backslash int_K\, S^{-1}(y)\} \cap \mathfrak{D}$ is compact for each $y \in K$. Hence $\bigcap_{y \in K}(\{K \backslash int_K\, S^{-1}(y)\} \cap \mathfrak{D}) \neq \emptyset$ and therefore $\bigcap_{y \in K}\{K \backslash int_K\, S^{-1}(y)\} \neq \emptyset$, which contradicts to our condition (b). Thus T has a fixed point. ∎

The following result generalizes Corollary 1 in [8].

Corollary 2.1. *Let K be a non-empty convex subset of a Hausdorff topological vector space X, and let $S, T : K \to 2^K$ be two multifunctions. Assume that*

(a) *for each $x \in K$, $co\big(S(x)\big) \subseteq T(x)$ and $S(x)$ is non-empty,*
(b) *$K = \bigcup\{int_K\, S^{-1}(y) : y \in K\}$,*
(c) *there exists a point $y_0 \in K$ such that $\{K \backslash int_K\, S^{-1}(y_0)\}$ is either empty or compact.*

Then there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

Proof. Take $B_0 = B_1 = \{y_0\}$ in Theorem 2.1. ∎

We note that the condition (b) of Theorems B, 2.1 and Corollary 2.1 is equivalent to the following condition of Tarafdar [7]:

(b′) for each $y \in K$, $S^{-1}(y) = \{x \in K : y \in S(x)\}$ contains a relatively open subset O_y of K (O_y could be empty) such that $\bigcup_{y \in K} O_y = K$.

3. Coincidence Theorems

The following coincidence theorem can be easily derived from Theorem 2.1.

Theorem 3.1. *Let K be a non-empty convex subset of a Hausdorff topological vector space X, and let $\Phi, \Psi : K \to 2^K$ be two multifunctions. Assume that the following conditions hold.*

(a) *For each $x \in K, \Psi^{-1}(\Phi(x))$ is non-empty and convex.*
(b) $K = \bigcup\{int_K\, \Phi^{-1}(\Psi(y)) : y \in K\}$.
(c) *There exists a non-empty subset B_0 of K such that B_0 is contained in a compact convex subset B_1 of K and the set $\mathfrak{D} = \bigcap\{K \backslash int_K\, \Phi^{-1}(\Psi(y)) : y \in B_0\}$ is either empty or compact.*

Then there exists $x_0 \in K$ such that $\Phi(x_0) \cap \Psi(x_0) \neq \emptyset$.

Proof. By taking $S \equiv \Psi^{-1} \circ \Phi$ in Theorem 2.1 for $S \equiv T$ and $S(x)$ is convex for all $x \in K$, we get the conclusion. ■

Corollary 3.1. *Let K be a non-empty convex subset of a Hausdorff topological vector space X, and let $\Phi, \Psi : K \to 2^K$ be two multifunctions. Assume that the following conditions hold.*

(a) *For each $x \in K, \Psi^{-1}(\Phi(x))$ is non-empty and convex.*
(b) $K = \bigcup\{int_K\, \Phi^{-1}(\Psi(y)) : y \in K\}$.
(c) *There exists a point $y_0 \in K$ such that $\{K \backslash int_K\, \Phi^{-1}(\Psi(y_0))\}$ is either empty or compact.*

Then there exists $x_0 \in K$ such that $\Phi(x_0) \cap \Psi(x_0) \neq \emptyset$.

Proof. Taking $B_0 = B_1 = \{y_0\}$ in Corollary 2.1. ■

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