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EXISTENCE AND DUALITY OF IMPLICIT VECTOR VARIATIONAL PROBLEMS*

Qamrul Hasan Ansari,[1] Xiao Qi Yang,[2] and Jen-Chih Yao[3]

[1]Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India
E-mail: qhansari@sancharnet.in
[2]Department of Applied Mathematics, Hong Kong Polytechnic University, Kowloon, Hong Kong
E-mail: mayangxq@polyu.edu.hk
[3]Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 804, Taiwan
E-mail: yaojc@math.nsysu.edu.tw

ABSTRACT

In this paper, we consider implicit vector variational problems which contain vector equilibrium problems and vector variational inequalities as special cases. The existence of solutions of implicit vector variational problems and vector equilibrium problems have been established. As a special case, we derive some existence results for a solution of vector variational inequalities. We also study the duality of implicit vector variational problems and discuss the relationship between solutions of dual and primal problems. Our results on duality contains known results in the literature as special cases.

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1. INTRODUCTION

The duality theory has been shown to be useful in mathematical economics, mechanics, numerical analysis and calculus of variations (see for example [8]). In 1972, Mosco [11] gave the dual form of a classical variational inequality [10] and proved its equivalence with the primal form. It has been extended by Yang [16] for vector variational inequalities, that is, variational inequalities for vector-valued operators. For further details on vector variational inequalities, we refer to [9] and references therein. Dolcetta and Matzeu [7] considered a more general problem known as implicit variational problem which includes variational and quasi-variational inequalities, fixed point and saddle points, Nash equilibria of non-cooperative games as special cases. They also studied its duality and applications. The existence of solution of this problem was studied by Mosco [12].

In this paper, we consider implicit vector variational problems, that is, implicit variational problems for vector-valued bifunctions, which contain vector equilibrium problems and vector variational inequalities [9] as special cases. We first establish the existence of solutions of implicit vector variational problems and vector equilibrium problems. As a special case, we then derive some existence results for a solution of vector variational inequalities. We also study the duality of implicit vector variational problems and discuss the relationships between solutions of dual and primal problems. Our results on duality contains known results in the literature as special cases.

2. PRELIMINARIES

Let Y be a topological vector space. A subset C of Y is called a *cone* if, $\lambda C \subseteq C$, for all $\lambda \geq 0$. It is easy to see that if a cone is convex then we further have $C + C = C$. A cone C is called *pointed* if, $C \cap (-C) = \{0\}$. Also, a cone C is called *proper* if, $C \neq Y$. Let C be a closed and convex cone in Y with non-empty interior, say int $C \neq \emptyset$. Given $x, y \in Y$, we consider the following (partial) ordering relations:

$$\begin{array}{ll} x \leq_C y \Leftrightarrow y - x \in C; & x \not\leq_C y \Leftrightarrow y - x \notin C; \\ x \geq_C y \Leftrightarrow x - y \in C; & x \not\geq_C y \Leftrightarrow x - y \notin C; \\ x \leq_{\text{int} C} y \Leftrightarrow y - x \in \text{int}\, C; & x \not\leq_{\text{int}\, C} y \Leftrightarrow y - x \notin \text{int}\, C; \\ x \geq_{\text{int} C} y \Leftrightarrow x - y \in \text{int}\, C; & x \not\geq_{\text{int}\, C} y \Leftrightarrow x - y \notin \text{int}\, C. \end{array}$$

For two given subsets A and B of Y, the following (partial) ordering relationships on sets are defined as follows:

$$A \leq_C B \Leftrightarrow x \leq_C y, \quad \text{for all } x \in A, y \in B;$$

$$A \not\leq_C B \Leftrightarrow x \not\leq_C y, \quad \text{for all } x \in A, y \in B;$$

$$A \leq_{\text{int}C} B \Leftrightarrow x \leq_{\text{int}C} y, \quad \text{for all } x \in A, y \in B;$$
$$A \not\leq_{\text{int}C} B \Leftrightarrow x \not\leq_{\text{int}C} y, \quad \text{for all } x \in A, y \in B.$$

A topological vector space Y with a closed and convex cone C which induces the (partial) ordering defined as above is called an *ordered topological vector space* and it is denoted by (Y, C).

Let X be a topological vector space, K a non-empty and convex subset of X and (Y, C) an ordered topological vector space. Given two bifunctions $\varphi, g: K \times K \to Y$, we consider the following *implicit vector variational problem* (for short, IVVP):

Find $\bar{x} \in K$ such that

$$\varphi(\bar{x}, \bar{x}) + g(\bar{x}, \bar{x}) \not\geq_{\text{int}C} \varphi(\bar{x}, y) + g(\bar{x}, y), \quad \text{for all } y \in K. \tag{1}$$

When $K = X$, $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, IVVP reduces to the problem of finding $\bar{x} \in X$ such that

$$\varphi(\bar{x}, \bar{x}) + g(\bar{x}, \bar{x}) \leq \varphi(\bar{x}, y) + g(\bar{x}, y), \quad \text{for all } y \in X, \tag{2}$$

which is known as the *implicit variational problem* (for short, IVP). It includes variational and quasi-variational inequalities, fixed point and saddle points, Nash equilibria of non-cooperative games as special cases. The existence of solution of IVP was studied by Mosco [12], while Dolcetta and Matzeu [7] discussed its duality and applications.

When $\varphi(x, x) = g(x, x) = 0$, for all $x \in K$, IVVP reduces to the following *vector equilibrium problem* (for short, VEP) considered by Ansari [1], Bianchi et al. [3] and Tan and Tinh [14]:

Find $\bar{x} \in K$ such that

$$\varphi(\bar{x}, y) + g(\bar{x}, y) \not\leq_{\text{int}C} 0, \quad \text{for all } y \in K. \tag{3}$$

Let $L(X, Y)$ be the space of all continuous linear operators from X to Y and $p, q: K \to L(X, Y)$ nonlinear operators. We denote by $\langle l, x \rangle$ the evaluation of $l \in L(X, Y)$ at $x \in X$. When $\varphi(x, y) = \langle p(x), y - x \rangle$ and, $g(x, y) = \langle q(x), y - x \rangle$, for all $x, y \in K$, VEP reduces to the following problem:

Find $\bar{x} \in K$ such that

$$\langle p(\bar{x}) + q(\bar{x}), y - \bar{x} \rangle \not\leq_{\text{int}C} 0, \quad \text{for all } y \in K. \tag{4}$$

It is known as the *strongly nonlinear vector variational inequality problem* (for short, SNVVIP) considered and studied by Ansari [1]. For $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, SNVVIP was studied by Mosco [12].

Now we mention some notation and results which will be used in the sequel.

Let K be a non-empty and convex subset of X. A function $f: K \to Y$ is called *C-lower semicontinuous* [3] on K if, for all $\gamma \in Y$, the (lower level) set

$$L(\gamma) = \{x \in K: f(x) \not\geq_{\text{int}C} \gamma\}$$

is closed in K. f is called *C-upper semicontinuous* [3] *on* K if, for all $\gamma \in Y$, the (upper level) set

$$U(\gamma) = \{x \in K: f(x) \not\leq_{\text{int}C} \gamma\}$$

is closed in K.

A function $f: K \to Y$ is called *C-continuous* on K if, it is both C-upper semicontinuous and C-lower semicontinuous.

We recall that a function $f: K \to Y$ is *upper semicontinuous with respect to C at a point* $x^* \in K$ [14] if, for any neighborhood V of $f(x^*)$ in Y, there exists a neighborhood U of x^* in X such that

$$f(x) \in V - C, \quad \text{for all } x \in U \cap K.$$

Furthermore, f is *upper semicontinuous with respect to C on K* if, it is upper semicontinuous with respect to C at each $x \in K$.

On the lines of the proof of Lemma 2.3 in [3], it is easy to show that a function $f: K \to Y$ is C-upper semicontinuous on K if and only if it is upper semicontinuous with respect to C.

If the functions $f_1, f_2: K \to Y$ are C-upper semicontinuous, then it can be easily proved that the function $f_1 + f_2$ is also C-upper semicontinuous (for C-lower semicontinuous case, see [3, pp. 531–532]).

A function $f: K \to Y$ is called *C-quasiconvex* [3] if, for all $\gamma \in Y$, the set

$$\{x \in K: f(x) \leq_C \gamma\}$$

is convex. f is called *C-quasiconcave* if, for all $\gamma \in Y$, the set

$$\{x \in K: f(x) \geq_C \gamma\}$$

is convex.

If f is C-quasiconvex (C-quasiconcave) then the set

$$\{x \in K: f(x) \leq_{\text{int}C} \gamma\} \ (\{x \in K: f(x) \geq_{\text{int}C} \gamma\}, \text{ respectively})$$

is also convex (see, for example [3]).

A bifunction $g: K \times K \to Y$ is called *C-diagonally convex* if, for any finite subset $\{x_1, \ldots, x_n\} \subset K$ and any $x_0 = \sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_i \geq 0$ for all $i=1, \ldots, n$ and $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$g(x_0, x_0) \not\geq_{\text{int}C} \sum_{i=1}^{n} \lambda_i g(x_0, x_i).$$

Similarly, g is called *C-diagonally concave* if, $-g$ is C-diagonally convex.

A bifunction $g: K \times K \to Y$ is called *strongly C-diagonally convex* if, for any finite subset $\{x_1, \dots, x_n\} \subset K$ and any $x_0 = \sum_{i=1}^n \lambda_i x_i$, where $\lambda_i \geq 0$ for all $i = 1, \dots, n$ and $\sum_{i=1}^n \lambda_1 = 1$, we have

$$g(x_0, x_0) \leq_C \sum_{i=1}^n \lambda_i g(x_0, x_i).$$

Similarly, g is called *strongly C-diagonally concave* if, $-g$ is strongly C-diagonally convex.

Remark 2.1. (a) If $\varphi: K \times K \to Y$ is C-diagonally convex (concave) and $g: K \times K \to Y$ is strongly C-diagonally convex (concave), then $\varphi + g$ is also C-diagonally convex (concave).
(b) When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the definition of C-diagonal convexity (concavity) reduces to the definition of diagonal convexity (concavity) [17] of a function.

Let A be a set, we shall denote by 2^A ($\Pi(A)$) the family of all subsets (respectively, non-empty subsets) of A and by $\mathcal{F}(A)$ the family of all non-empty finite subsets of A. If A and B are subsets of a topological space such that $B \subseteq A$, we shall denote by $\text{int}_A B$ ($\text{cl}_A B$) the interior (closure, respectively) of B in A. If A is a subset of a vector space, $co(A)$ denotes the convex hull of A.

A multi-valued map $T: K \to \Pi(K)$ is called a *KKM-map* if, for every finite subset $\{x_1, \dots, x_n\}$ of K, $co(\{x_1, \dots, x_n\}) \subset \cup_{i=1}^n T(x_i)$.

The following result of Chowdhury and Tan [5] will be used in proving the existence of a solution of IVVP.

Lemma 2.1. [5] Let K be a non-empty and convex subset of a topological vector space X. Let $T: K \to \Pi(K)$ be a KKM-map such that

(a) $\text{cl}_K T(\tilde{y})$ is compact for some $\tilde{y} \in K$,
(b) for each $A \in \mathcal{F}(K)$ with $\tilde{y} \in A$ and each $y \in co(A)$, $T(y) \cap co(A)$ is closed in $co(A)$, and
(c) for each $A \in \mathcal{F}(K)$ with $\tilde{y} \in A$,

$$\left(\text{cl}_K\left(\bigcap_{y \in co(A)} T(y)\right)\right) \bigcap co(A) = \left(\bigcap_{y \in co(A)} T(y)\right) \bigcap co(A).$$

Then $\bigcap_{y \in K} T(y) \neq \emptyset$.

3. EXISTENCE RESULTS

Now we are ready to derive the following existence result for a solution to IVVP.

Theorem 3.1. Let K be a non-empty and convex subset of a topological vector space X, (Y, C) an ordered topological vector space with C proper, and $\varphi, g: K \times K \to Y$ bifunctions. Assume that the following conditions hold.

(i) φ is C-diagonally convex.
(ii) g is strongly C-diagonally convex.
(iii) For each $A \in \mathscr{F}(K)$, φ and g are continuous on $co(A)$.
(iv) For each $A \in \mathscr{F}(K)$, and each $x, y \in co(A)$ and every net $\{x_\alpha\}_{\alpha \in \Gamma}$ in K converging to x with

$$\varphi(x_\alpha, x_\alpha) + g(x_\alpha, x_\alpha) \not\geq_{\text{int}\, C} \varphi(x_\alpha, \lambda y + (1-\lambda)x) + g(x_\alpha, \lambda y + (1-\lambda)x),$$

for all $\alpha \in \Gamma$ and all $\lambda \in [0, 1]$, we have

$$\varphi(x, x) + g(x, x) \not\geq_{\text{int}\, C} \varphi(x, y) + g(x, y).$$

(v) There exist a non-empty, closed and compact subset B of K and $\tilde{y} \in B$ such that

$$\varphi(x, x) + g(x, x) \geq_{\text{int}\, C} \varphi(x, \tilde{y}) + g(x, \tilde{y}), \quad \text{for all } x \in K \backslash B.$$

Then IVVP has a solution $\bar{x} \in B$.

Proof. For each $y \in K$, we define a multi-valued map $T: K \to \Pi(K)$ by

$$T(y) = \{x \in K: \varphi(x, x) + g(x, x) \not\geq_{\text{int}C} \varphi(x, y) + g(x, y)\}.$$

Then clearly for each $y \in K$, $T(y)$ is non-empty, since $y \in T(y)$ by the properness of C. By (i) and (ii), T is a KKM-map. We also have

(a) $T(\tilde{y}) \subset B$, so that $\text{cl}_K T(\tilde{y}) \subset \text{cl}_K B = B$ and hence $\text{cl}_K T(\tilde{y})$ is compact in K;
(b) For each $A \in \mathscr{F}(K)$ with $\tilde{y} \in A$ and $y \in co(A)$,

$$T(y) \cap co(A) = \{x \in co(A): \varphi(x, x) + g(x, x) \not\geq_{\text{int}C} \varphi(x, y) + g(x, y)\}$$

is closed in co(A) by (iii);
(c) For each $A \in \mathscr{F}(K)$ with $\tilde{y} \in A$, let $x \in (\text{cl}_K(\bigcap_{y \in co(A)} T(y))) \cap co(A)$, then $x \in co(A)$ and there is a net $\{x_\alpha\}_{\alpha \in \Gamma}$ in $\bigcap_{y \in co(A)} T(y)$ such that $x_\alpha \to x$. For each $y \in co(A)$, since $\lambda y + (1-\lambda)x \in co(A)$, for all $\lambda \in [0, 1]$, we have $x_\alpha \in T(\lambda y + (1-\lambda)x)$, for all $\alpha \in \Gamma$ and all $\lambda \in [0, 1]$. This implies that

$$\varphi(x_\alpha, x_\alpha) + g(x_\alpha, x_\alpha) \not\geq_{\text{int}C} \varphi(x_\alpha, \lambda y + (1-\lambda)x) + g(x_\alpha, \lambda y + (1-\lambda)x)$$

for all $\alpha \in \Gamma$ and all $\lambda \in [0, 1]$. By (v), we have

$$\varphi(x, x) + g(x, x) \not\geq_{\text{int}C} \varphi(x, y) + g(x, y).$$

It follows that $x \in (\bigcap_{y \in co(A)} T(y)) \cap co(A)$. Hence

$$\left(\text{cl}_K\left(\bigcap_{y \in co(A)} T(y)\right)\right) \cap co(A) = \left(\bigcap_{y \in co(A)} T(y)\right) \cap co(A).$$

By Lemma 2.1, we have $\bigcap_{y \in K} T(y) \neq \emptyset$. Hence there exists $\bar{x} \in \bigcap_{y \in K} T(y)$ and therefore

$$\varphi(\bar{x}, \bar{x}) + g(\bar{x}, \bar{x}) \not\geq_{\text{int}C} \varphi(\bar{x}, y) + g(\bar{x}, y), \quad \text{for all } y \in K,$$

from which it follows that $\bar{x}$ is a solution of IVVP. □

When $\varphi(x, y) = f(y)$ and $g(x, x) = 0$, for all $x, y \in K$, we have the following result.

Theorem 3.2. Let K be a non-empty and convex subset of a topological vector space $X, (Y, C)$ an ordered topological vector space with C proper, $f: K \to Y$ a function and $g: K \times K \to Y$ a bifunction. Assume that the following conditions hold.

(i) For each $x \in K, y \mapsto g(x, y) + f(y)$ is C-quasiconvex.
(ii) For each $A \in \mathscr{F}(K)$, f is C-lower semicontinuous on $co(A)$.
(iii) For each $A \in \mathscr{F}(K)$ and for each $y \in co(A), x \mapsto g(x, y)$ is C-upper semicontinuous on $co(A)$.
(iv) For each $A \in \mathscr{F}(K)$ and each $x, y \in co(A)$ and every net $\{x_\alpha\}_{\alpha \in \Gamma}$ in K converging to x with

$$f(x_\alpha) - f(\lambda y + (1 - \lambda)x) \not\geq_{\text{int}C} g(x_\alpha, \lambda y + (1 - \lambda)x),$$

for all $\alpha \in \Gamma$ and all $\lambda \in [0, 1]$, we have

$$f(x) - f(y) \not\geq_{\text{int}C} g(x, y).$$

(v) There exist a non-empty, closed and compact subset B of K and $\bar{y} \in B$ such that

$$f(x) - f(\bar{y}) \geq_{\text{int}C} g(x, \bar{y}), \quad \text{for all } x \in K \backslash B.$$

Then there exists $\bar{x} \in B$ such that

$$f(\bar{x}) - f(y) \not\geq_{\text{int}C} g(\bar{x}, y), \quad \text{for all } y \in K.$$

Proof. It is on the lines of the proof of Theorem 3.1. □

Remark 3.1. Theorem 3.2 can be viewed as an existence of Theorem 1 in [5] for vector-valued case.

We need the following result of Ding [6] in order to prove the existence of a solution to the vector variational inequalities.

Lemma 3.1. [6] Let X and Y be topological vector spaces and let $L(X,Y)$ be equipped with uniform convergence topology δ (see [6, pp. 79–81]). Then the bilinear form $\langle .,. \rangle : L(X,Y) \times X \to Y$ is continuous on $(L(X,Y),\delta) \times X$.

When $\varphi(x,y) = f(y)$ and $g(x,y) = \langle p(x), y - x \rangle$, for all $x, y \in K$, where $p: K \to L(X,Y)$ is a nonlinear operator, then we can easily derive the following result from Theorem 3.2.

Corollary 3.1. Let K be a non-empty and convex subset of a topological vector space X, (Y, C) an ordered topological vector space with C proper, $f: K \to Y$ a function and $p: K \to L(X,Y)$ a nonlinear operator. Let $L(X,Y)$ be equipped with uniform convergence topology. Assume that the following conditions hold.

(i) f is C-quasiconvex.
(ii) For each $A \in \mathcal{F}(K)$, f is C-lower semicontinuous on $co(A)$.
(iii) For each $A \in \mathcal{F}(K)$, p is continuous on $co(A)$.
(iv) For each $A \in \mathcal{F}(K)$ and each $x, y \in co(A)$ and every net $\{x_\alpha\}_{\alpha\in\Gamma}$ in K converging to x with

$$f(x_\alpha) - f(\lambda y + (1-\lambda)x) \not\geq_{\text{int}\,C} \langle p(x_\alpha), (\lambda y + (1-\lambda)x) - x_\alpha \rangle,$$

for all $\alpha \in \Gamma$ and all $\lambda \in [0,1]$, we have

$$f(x) - f(y) \not\geq_{\text{int}C} \langle p(x), y - x \rangle.$$

(v) There exist a non-empty, closed and compact subset B of K and $\tilde{y} \in B$ such that

$$f(x) - f(\tilde{y}) \not\geq_{\text{int}C} \langle p(x), \tilde{y} - x \rangle, \quad \text{for all } x \in K \backslash B.$$

Then there exists $\bar{x} \in B$ such that

$$f(\bar{x}) - f(y) \not\geq_{\text{int}C} \langle p(\bar{x}), y - \bar{x} \rangle, \quad \text{for all } y \in K.$$

Remark 3.2. In Theorems 3.1 and 3.2 and Corollary 3.1, X need not be Hausdorff.

In order to establish some existence results for a solution to VEP, we need the following result due to Ansari and Yao [2].

Lemma 3.2. [2] Let K be a non-empty and convex subset of a Hausdorff topological vector space X and $S, T : K \rightarrow 2^K$ be two multi-valued maps. Assume that the following conditions hold.

(a) For each $x \in K, co(S(x)) \subset T(x)$ and $S(x)$ is non-empty.
(b) $K = \cup\{\text{int}_K S^{-1}(y) : y \in K\}$.
(c) If K is not compact, assume that there exist a non-empty, compact and convex subset B of K and a non-empty and compact subset D of K such that for each $x \in K\backslash D$, there exists $\tilde{y} \in B$ such that $x \in \text{int}_K S^{-1}(\tilde{y})$.

Then there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

Now we can state and prove the following existence result for a solution to VEP.

Theorem 3.3. Let K be a non-empty and convex subset of a Hausdorff topological vector space X, (Y, C), an ordered topological vector space with C proper and $\varphi, g: K \times K \rightarrow Y$ bifunctions. Assume that the following conditions hold.

(i) For each $x, y \in K, \varphi(x, y) + g(x, y) \leq_{\text{int}C} 0$ implies $g(y, x) \geq_{\text{int}C} 0$.
(ii) For each fixed $y \in K, x \mapsto g(x, y)$ is C-quasiconcave and C-upper semicontinuous on K.
(iii) For each fixed $y \in K, x \mapsto g(x, y)$ is C-upper semicontinuous on K.
(iv) If K is not compact, assume that there exist a non-empty, compact and convex subset B of K and a non-empty and compact subset D of K such that for each $x \in K\backslash D$, there exists $\tilde{y} \in B$ such that

$$\varphi(x, \tilde{y}) + g(x, \tilde{y}) \leq_{\text{int}C} 0.$$

Then VEP has a solution.

Proof. Assume that the conclusion of this theorem is not true. Then for each $x \in K$, the set

$$\{y \in K: \varphi(x, y) + g(x, y) \leq_{\text{int}C} 0\} \neq \emptyset.$$

We define two multi-valued maps $S, T: K \rightarrow 2^K$ by

$$S(x) = \{y \in K: \varphi(x, y) + g(x, y) \leq_{\text{int}C} 0\}$$

and

$$T(x) = \{y \in K: g(y, x) \geq_{\text{int}C} 0\}, \quad \text{for all } x \in K.$$

Then clearly for all $x \in K$, $S(x) \neq \emptyset$. Let $\{y_1, \ldots, y_n\}$ be a finite subset of $S(x)$ and $\lambda_i \geq 0$ for all $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$. Then

$$\varphi(x, y_i) + g(x, y_i) \leq_{\text{int}C} 0, \quad \text{for all } i.$$

By (i), we have

$$g(y_i, x) \geq_{\text{int}C} 0, \quad \text{for all } i.$$

Since $g(.,x)$ is C-quasiconcave, we have

$$g\left(\sum_{i=1}^n \lambda_i y_i, x\right) \geq_{\text{int}C} 0$$

and hence $\sum_{i=1}^n \lambda_i y_i, \in T(x)$ and therefore $co(S(x)) \subset T(x)$, for all $x \in K$.

Since $\varphi(., y)$ and $g(., y)$ are C-upper semicontinuous and so $\varphi(., y) + g(., y)$ is also C-upper semicontinuous. Therefore the complement of $S^{-1}(y)$ in K,

$$[S^{-1}(y)]^c = \{x \in K : \varphi(x, y) + g(x, y) \not\leq_{\text{int}C} 0\}$$

is closed in K. Hence $S^{-1}(y)$ is open in K. Since $S(x) \neq \emptyset$, for all $x \in K$, we have

$$K = \bigcup_{y \in K} S^{-1}(y) = \bigcup_{y \in K} \text{int}_K S^{-1}(y).$$

By (iv), for each $x \in K \setminus D$, there exists $\tilde{y} \in B$ such that

$$x \in S^{-1}(\tilde{y}) = \text{int}_K S^{-1}(\tilde{y}).$$

Hence all the conditions of Lemma 3.2 are satisfied and therefore there exists $x_0 \in K$ such that $x_0 \in T(x_0)$, that is, $g(x_0, x_0) \geq_{\text{int}C} 0$, which contradicts to the fact that $0 \notin \text{int}\, C$ because C is proper. This completes the proof. □

From Theorem 3.3 we can derive the following existence result for a solution to SNVVIP.

Corollary 3.2. Let K be a non-empty and convex subset of a Hausdorff topological vector space X, (Y, C) an ordered topological vector space with C proper and $p, q : K \to L(X, Y)$ two continuous nonlinear operators. Let $L(X, Y)$ be equipped with uniform convergence topology. Assume that the following conditions hold.

(i) For each $x, y \in K$, $\langle p(x) + q(x), y - x\rangle \leq_{\text{int}C} 0$ implies $\langle q(y), x - y\rangle \geq_{\text{int}C} 0$;

(ii) For each $y \in K$, the function $x \mapsto \langle q(x), y - x\rangle$ is C-quasiconvex;

(iii) If K is not compact, assume that there exist a non-empty, compact and convex subset B of K and a non-empty and compact subset D of K such that for each $x \in K\backslash D$, there exists $\tilde{y} \in B$ such that

$$\langle p(x) + q(x), x - \tilde{y} \rangle \leq_{\text{int}C} 0.$$

Then SNVVIP has a solution.

4. DUALITY OF IMPLICIT VECTOR VARIATIONAL PROBLEMS

Let X be a Banach space and (Y, C) be an ordered Banach space with the proper, closed and convex cone C such that $\text{int}\, C \neq \emptyset$. Given two bifunctions $\varphi, g: X \times X \to Y$, then we consider the following *implicit vector variational problem:*

(P) Find $\bar{x} \in X: \varphi(\bar{x}, \bar{x}) + g(\bar{x}, \bar{x}) \not\geq_{\text{int}\, C} \varphi(\bar{x}, y) + g(\bar{x}, y)$, for all $y \in X$.

Let $g: X \to Y$ and $y \in X$. If a linear operator $l \in L(X, Y)$ satisfies

$$\langle l, h \rangle \not\geq_{\text{int}\, C} g(h + y) - g(y), \quad \text{for all } h \in X,$$

then l is said to be a *weak subgradient of g at y* [16]. The set of all weak subgradients of g at y is denoted by $\partial^w g(y)$ and g is said to be *weakly subdifferentiable* at y [16] if, $\partial^w g(y) \neq \emptyset$.

Let $A \subset Y$. Denote

$$\text{Sup}_{\text{int}C} A = \{u \in \text{cl}A : (A - \{u\}) \cap \text{int}C = \emptyset\},$$

where cl$A$ is the closure of A. In the case that $Y = \mathbb{R}$ and $C = [0, \infty)$, we have

$$\text{Sup}_{\text{int}C} A = \begin{cases} \text{Sup}A, & \{u \in \text{cl}A : (A - \{u\}) \cap \text{int}\, C = \emptyset\} \neq \emptyset \\ \infty, & \{u \in \text{cl}A : (A - \{u\}) \cap \text{int}\, C = \emptyset\} = \emptyset \end{cases}$$

Let $g: X \to Y$ and $l \in L(X, Y)$. The *vector conjugate function* [16], denoted as g^*_{sup}, of g at y is defined by

$$g^*_{\text{sup}}(l) = \text{Sup}_{\text{int}C}\{\langle l, y \rangle - g(y) : y \in x\}.$$

Let $y \in X$. The *vector biconjugate function*, denoted as g^{**}_{sup} of g at y is defined by

$$g^{**}_{\text{sup}} = \text{Sup}_{\text{int}C} \bigcup \{\langle l, y \rangle - g^*_{\text{sup}}(l) : \ l \in L(X, Y)\}.$$

Note that both $g^*_{\sup}$ and $g^{**}_{\sup}$ are multi-valued maps and $g^*_{\sup}: L(X, Y) \to 2^Y$, $g^{**}_{\sup}: X \to 2^Y$. Throughout this section, we assume that $g^*_{\sup}(l) \neq \emptyset$ and $g^{**}_{\sup}(y) \neq \emptyset$.

Let $g: X \to Y$ and $y \in X$. g is said to be *externally stable at* y if, $g(y) \in g^{**}_{\sup}(y)$.

The external stability was introduced in [13] when the vector conjugate function is defined via the set of efficient points.

Lemma 4.1. Let $g : X \to Y$ and $y \in X$. Then

$$l \in \partial^w g(y) \quad \Leftrightarrow \quad \langle l, y\rangle - g(y) \in g^*_{\sup}(l). \tag{5}$$

Proof. It follows from the definitions of the vector conjugate function and the subgradient that $l \in \partial^w g(y)$ if and only if

$$\langle l, h\rangle \not\geq_{\text{int}C} g(h + y) - g(y), \quad \text{for all } h \in X,$$

equivalently,

$$\langle l, h + y\rangle - g(h + y) \not\geq_{\text{int}C} \langle l, y\rangle - g(y), \quad \text{for all } h \in X,$$

if and only if $\langle l, y\rangle - g(y) \in g^*_{\sup}(l)$ as $y \in X$. □

For a fixed $y \in X, \varphi(y, .) : z \mapsto \varphi(y, z)$ is *convex* [13] if,

$$\varphi(y, \lambda z_1 + (1 - \lambda)z_2) \leq_C \lambda\varphi(y, z_1) + (1 - \lambda)\varphi(y, z_2),$$

for all $z_1, z_2 \in X$ and $\lambda \in (0, 1)$.

It is shown in [4] that if $\varphi(y, .): z \mapsto \varphi(y, z)$ is convex, then $\partial^w\varphi(y, z) \neq \emptyset$, where $\partial^w\varphi(y, z)$ denotes the weak subdifferential of φ with respect to its second component.

We now define the dual problem of (P) as follows.

(D) Find $\bar{x} \in X$, $-l^* \in \partial^w g(\bar{x}, \bar{x})$ and $\bar{y} \in \varphi^*_{\sup}(\bar{x}, l^*)$ satisfying $\langle l^*, \bar{x}\rangle - \varphi(\bar{x}, \bar{x}) = \bar{y}$ such that

$$\bar{y} - \langle l^*, \bar{x}\rangle \not\geq_{\text{int}C} \varphi^*_{\sup}(\bar{x}, l) - \langle l, \bar{x}\rangle, \quad \text{for all } l \in L(X, Y).$$

(D) is called the *dual implicit vector variational problem* and $(\bar{x}, l^*)$ is called a solution of (D). The following two results show the relationships between solutions of the implicit vector variational problem (P) and the dual implicit vector variational problem (D).

Theorem 4.1. Assume that $g(y,.)$ and $\varphi(y,.)$ are convex for each fixed $y \in X$. If $\bar{x}$ is a solution of (P) and $\varphi(\bar{x},.) : x \mapsto \varphi(\bar{x}, x)$ is externally stable, then there exists $l^* \in L(X, Y)$ such that $(\bar{x}, l^*)$ is a solution of (D).

Proof. Let $\bar{x}$ be a solution of (P). Then

$$\varphi(\bar{x},\bar{x}) + g(\bar{x},\bar{x}) \not\geq_{\text{int}C} \varphi(\bar{x},z) + g(\bar{x},z), \quad \text{for all } z \in X.$$

Let 0^* be the zero operator from X to Y. Then

$$\langle 0^*, h\rangle \not\geq_{\text{int}C} [\varphi(\bar{x},z) + g(\bar{x},z)] - [\varphi(\bar{x},\bar{x}) + g(\bar{x},\bar{x})], \quad \text{for all } z \in X.$$

By the definition of $\partial^w g(y,z)$, we have

$$0^* \in \partial^w(\varphi(\bar{x},\bar{x}) + g(\bar{x},\bar{x})).$$

It follows from [15] that

$$\partial^w(\varphi(\bar{x},\bar{x}) + g(\bar{x},\bar{x})) \subseteq \partial^w\varphi(\bar{x},\bar{x}) + \partial^w g(\bar{x},\bar{x}).$$

Hence

$$0^* \in \partial^w\varphi(\bar{x},\bar{x}) + \partial^w g(\bar{x},\bar{x}),$$

or equivalently, there exists $l^* \in L(X, Y)$ such that

$$l^* \in \partial^w\varphi(\bar{x},\bar{x}) \cap (-\partial^w g(\bar{x},\bar{x})).$$

Then from (5), we obtain

$$\begin{aligned} &\langle l^*, \bar{x}\rangle - \varphi(\bar{x},\bar{x}) \in \varphi^*_{\text{sup}}(\bar{x}, l^*), \\ &\langle -l^*, \bar{x}\rangle - g(\bar{x},\bar{x}) \in g^*_{\text{sup}}(\bar{x}, -l^*). \end{aligned} \tag{6}$$

As $\varphi(\bar{x},.) : x \mapsto \varphi(\bar{x}, x)$ is externally stable,

$$\varphi(\bar{x},\bar{x}) \in \varphi^{**}_{\text{sup}}(\bar{x},\bar{x}) = \text{Sup}_{\text{int}C}\left\{\langle l, \bar{x}\rangle - \varphi^*_{\text{sup}}(\bar{x}, l) : l \in L(X, Y)\right\}.$$

Thus

$$\varphi(\bar{x},\bar{x}) \not\geq_{\text{int}C} \langle l, \bar{x}\rangle - \varphi^*_{\text{sup}}(\bar{x}, l), \quad \text{for all } l \in L(X, Y).$$

From (6), there exists $\bar{y} \in \varphi^*_{\text{sup}}(\bar{x}, l^*)$ such that

$$\langle l^*, \bar{x}\rangle - \varphi(\bar{x},\bar{x}) = \bar{y},$$

that is,

$$\varphi(\bar{x},\bar{x}) = \langle l^*, \bar{x}\rangle - \bar{y}.$$

So

$$\langle l^*, \bar{x}\rangle - \bar{y} \not\leq_{\text{int}C} \langle l(\bar{x})\rangle - \varphi^*_{\sup}(\bar{x}, l), \quad \text{for all } l \in L(X, Y).$$
$$\bar{y} - \langle l^*, \bar{x}\rangle \not\geq_{\text{int}C} \varphi^*_{\sup}(\bar{x}, l) - \langle l, \bar{x}\rangle, \quad \text{for all } l \in L(X, Y).$$

Thus $(\bar{x}, l^*)$ is a solution of (D). □

Theorem 4.2. Assume that $g(y,.)$ and $\varphi(y,.)$ are convex for each fixed $y \in X$. If $(\bar{x}, l^*)$ is a solution of (D) and

$$\partial^w(\varphi(\bar{x}, \bar{x}) + g(\bar{x}, \bar{x})) = \partial^w \varphi(\bar{x}, \bar{x}) + \partial^w g(\bar{x}, \bar{x}),$$

then $\bar{x}$ is a solution of (P).

Proof. This is obtained by inverting the reasoning in the proof of Theorem 4.1 step by step. □

It can be shown that if the convex cone C has the following property:

$$a \not\leq_{\text{int}C} b \;\Rightarrow\; a \geq_C b, \tag{7}$$

then we can prove that

$$\partial^w(\varphi(\bar{x}, \bar{x}) + g(\bar{x}, \bar{x})) = \partial^w \varphi(\bar{x}, \bar{x}) + \partial^w g(\bar{x}, \bar{x}). \tag{8}$$

In fact, let $l_1 \in \partial^w \varphi(\bar{x}, \bar{x}), l_2 \in \partial^w g(\bar{x}, \bar{x})$. Then by (7) for any $z \in X$, We have

$$\varphi(\bar{x}, \bar{x} + z) \geq_C \varphi(\bar{x}, \bar{x}) + \langle l_1, z\rangle$$

and

$$g(\bar{x}, \bar{x} + z) \geq_C g(\bar{x}, \bar{x}) + \langle l_2, z\rangle.$$

Thus by convexity of C

$$\varphi(\bar{x}, \bar{x} + z) + g(\bar{x}, \bar{x} + z) \geq_C \varphi(\bar{x}, \bar{x}) + g(\bar{x}, \bar{x}) + \langle l_1 + l_2, z\rangle, \text{ for all } z \in X.$$

So $l_1 + l_2 \in \partial^w(\varphi(\bar{x}, \bar{x}) + g(\bar{x}, \bar{x}))$. Noting the inclusion in other direction holds (see [15]), the equality (8) holds.

Remark 4.1. Theorems 4.1 and 4.2 are extensions of Theorem 1 in [7] for vector-valued bifunctions.

Indeed, let $Y = \mathbb{R}$ and $C = \mathbb{R}_+$. Then (D) is reduced to find $\bar{x} \in X$, $l^* \in X^*$ such that $-l^* \in \partial g(\bar{x}, \bar{x})$ and

$$\varphi^*(\bar{x}, l^*) - \langle l^*, \bar{x}\rangle \leq \varphi^*(\bar{x}, l) - \langle l, \bar{x}\rangle, \quad \text{for all } l \in X^*,$$

where $\partial g(\bar{x}, \bar{x})$ and $\varphi^*(\bar{x}, l^*)$ are convex subdifferential and convex conjugate functions, respectively [8]. This is a dual problem of IVP which was studied by Dolcetta and Matzcu [7].

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