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Existence results for Stampacchia and Minty type implicit variational inequalities with multivalued maps☆

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Abstract

Stampacchia and Minty type generalized implicit variational inequality problems are considered. We extended the notion of dense pseudomonotonicity for multivalued maps and established several existence results for solutions of these problems in the setting of segment-dense sets. We also studied the existence of solutions of Minty type generalized implicit quasi-variational inequality problems. Some particular cases are also studied. It is shown that our results contain several existing results in the literature as special cases.

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1. Introduction and formulations

Let X and Y be real Hausdorff topological vector spaces and let $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{R}$ be a continuous bilinear form. We denote by 2^X the family of all subsets of X. Let C be a nonempty subset of Y and $T : C \to 2^X$ be a multivalued map with nonempty values. Following the terminology of Giannessi [9] (see also [14]), we recall the following Minty and Stampacchia variational inequality problems:

The *Minty variational inequality problem* is the following:

$$\text{(MVIP)}\qquad \begin{cases} \text{Find } \bar{u} \in C \text{ such that for all } v \in C \text{ and for all } y \in T(v), \\ \langle y, v - \bar{u}\rangle \geqslant 0. \end{cases}$$

That is, $\bar{u} \in C$ is a solution of (MVIP) if and only if

$$\inf\langle T(v), v - \bar{u}\rangle := \inf_{y \in T(v)} \langle y, v - \bar{u}\rangle \geqslant 0 \quad \text{for all } v \in C.$$

The Stampacchia variational inequality problem can be formulated as follows:

$$\text{(SVIP)}\qquad \begin{cases} \text{Find } \bar{u} \in C \text{ such that there exists } \bar{x} \in T(\bar{u}) \text{ satisfying} \\ \langle \bar{x}, v - \bar{u}\rangle \geqslant 0 \quad \text{for all } v \in C. \end{cases}$$

If the multivalued map T has compact values, then (SVIP) reduces to find $\bar{u} \in C$ such that

$$\sup\langle T(\bar{u}), v - \bar{u}\rangle := \sup_{\bar{x} \in T(\bar{u})} \langle \bar{x}, v - \bar{u}\rangle \geqslant 0 \quad \text{for all } v \in C.$$

In the last decade these two problems were extensively studied by many researcher, see for example [1,2,7,10,11,14,20,19] and the references therein. It is well known that above mentioned problems play a vital role in nondifferentiable optimization problems, economics, game theory, mechanics etc; see for example [1,2,11,18,20] and the references therein.

Let $T : C \times C \to 2^X$ and $F : X \times C \times C \to 2^{\mathbb{R}}$ be multivalued maps with nonempty values. We consider the following *Stampacchia type generalized implicit variational inequality problem*:

$$\text{(SGIVIP)}\qquad \begin{cases} \text{Find } \bar{u} \in C \text{ such that} \\ \sup F(T(\bar{u}, \bar{u}), u, \bar{u}) \geqslant 0 \quad \text{for all } u \in C. \end{cases}$$

When F is a single-valued map, then (SGIVIP) reduces to the following *Stampacchia type implicit variational inequality problem:*

Let $f : X \times C \times C \to \mathbb{R}$ be a given function.

$$\text{(SIVIP)}\qquad \begin{cases} \text{Find } \bar{u} \in C \text{ such that} \\ \displaystyle\sup_{\bar{x} \in T(\bar{u}, \bar{u})} f(\bar{x}, u, \bar{u}) \geqslant 0 \quad \text{for all } u \in C. \end{cases}$$

When $Y = X^*$ (the topological dual of X) and $f(x, v, u) = \langle x, v - u\rangle$ for all $x \in X^*$ and $u, v \in C$, (SIVIP) reduces to the following *generalized variational inequality problem*:

$$\text{(GVIP)}\qquad \begin{cases} \text{Find } \bar{u} \in C \text{ such that} \\ \displaystyle\sup_{\bar{x} \in T(\bar{u}, \bar{u})} \langle \bar{x}, u - \bar{u}\rangle \geqslant 0 \quad \text{for all } u \in C. \end{cases}$$

Chen [6] established an existence result for a solution of (GVIP) for semi-monotone multivalued map, that is, multivalued map which is completely continuous in the first argument and monotone in the second argument. The semi-monotone operators occur in the theory of nonlinear elliptic operators in divergence form which are monotone only in those terms in the principal part, that is, the highest order terms. Kassay and Kolumbán [10] improved the results of Chen [6]. They considered the multivalued map T being pseudomonotone (see [19]) in the first argument and having certain kind of continuity conditions in the second argument. Such types of multivalued maps are called *semi-pseudomonotone*.

We note that (GVIP) contains many known variational inequalities for multivalued maps studied in [1,2,7,14,20,19] and references therein.

From the above special cases, it is clear that our (SGIVIP) and (SIVIP) are more general and unified one.

We also consider the following *Minty type generalized implicit quasi-variational inequality problem*:

$$\text{(MGIQVIP)} \quad \begin{cases} \text{Find } \bar{u} \in C \text{ such that } \bar{u} \in B(\bar{u}) \text{ and} \\ \inf F(T(v,\bar{u}), v, \bar{u}) \geqslant 0 \quad \text{for all } v \in A(\bar{u}), \end{cases}$$

where $A, B : C \to 2^C$ are multivalued maps with nonempty values.

When F is a single-valued map, then (MGIQVIP) becomes the following *Minty type implicit quasi-variational inequality problem*:

Let $f : X \times C \times C \to \mathbb{R}$ be a given function.

$$\text{(MIQVIP)} \quad \begin{cases} \text{Find } \bar{u} \in C \text{ such that } \bar{u} \in B(\bar{u}) \text{ and} \\ \inf\limits_{\bar{y} \in T(v,\bar{u})} f(\bar{y}, v, \bar{u}) \geqslant 0 \quad \text{for all } v \in A(\bar{u}). \end{cases}$$

When $Y = X^*$ (the topological dual of X) and $f(y, v, u) = \langle y, v - u\rangle$ for all $y \in X^*$ and $u, v \in C$, (MIQVIP) reduces to the following *Minty type generalized quasi-variational inequality problem*:

$$\text{(MGQVIP)} \quad \begin{cases} \text{Find } \bar{u} \in C \text{ such that } \bar{u} \in B(\bar{u}) \text{ and} \\ \inf\limits_{\bar{y} \in T(v,\bar{u})} \langle \bar{y}, v - \bar{u}\rangle \geqslant 0 \quad \text{for all } v \in A(\bar{u}). \end{cases}$$

When $A(u) = B(u) = C$ for all $u \in C$, then (MGIQVIP), (MIQVIP) and (MGQVIP) are called *Minty type generalized implicit variational inequality problem*, *Minty type implicit variational inequality problem* and *Minty type generalized variational inequality problem*, respectively.

In Section 2, we present basic definitions, notations and results which will be used in the sequel. Recently, Luc [17] introduced a weaker concept of pseudomonotonicity, called *dense pseudomonotonicity*, and extended the classical Hartman–Stampacchia theorem [12] for this concept. In Section 3, we extend the notion of dense pseudomonotonicity for multivalued maps and establish several existence results for a solution of (SGIVIP). As consequences of our results, we derive the existence results for a solution of (GVIP) under dense pseudomonotonicity assumption. The results of this section contain the results of Luc [17] and Kassay and Kolumbán [10] as special cases. In Section 4, we establish an existence result for a solution of (MGIQVIP) by using an equilibrium theorem of Lin et al. [16]. Finally, we show that the classical existence result concerning Minty variational inequalities with monotone operators is an easy consequence of our results. The results of this section also include the results of Lin et al. [16] as special cases.

2. Preliminaries

Let D be a nonempty subset of a topological vector space X. The interior of D, the closure of D and the convex hull of D are denoted by int D, $\overline{D}$ and co$D$, respectively. Throughout the paper, all topological spaces are assumed to be Hausdorff.

Definition 2.1. Let X and Y be topological spaces. A multivalued map $T : X \to 2^Y$ is said to be

(a) *upper semicontinuous* if for each closed set $D \subseteq Y$, the set $T^{+1}(D) = \{x \in X : T(x) \cap D \neq \emptyset\}$ is closed in X;
(b) *lower semicontinuous* if for each open set $D \subseteq Y$, the set $T^{+1}(D)$ is open in X;
(c) *continuous* if it is both upper and lower semicontinuous.

The following lemmas will be used in the sequel.

Lemma 2.1 (*Lin and Yu [15]*). *Let X and Y be topological spaces. Let $F : X \times Y \to 2^{\mathbb{R}}$ and $S : X \to 2^Y$ be multivalued maps with nonempty values and let $m(x) = \sup F(x, S(x))$ and $M(x) = \{y \in S(x) : m(x) \in F(x, y)\}$.*

(a) *If both F and S are lower semicontinuous, then m is also lower semicontinuous.*
(b) *If both F and S are upper semicontinuous with compact values, then m is also upper semicontinuous.*
(c) *If both F and S are continuous with compact values, then m is a continuous function and M is an upper semicontinuous and closed multivalued map.*

Lemma 2.2 (*Aubin and Cellina [4]*). *Let X and Y be topological spaces and let $T : X \to 2^Y$ be a multivalued map.*

(a) *If X is compact and T is upper semicontinuous with compact values, then $T(X)$ is compact.*
(b) *If Y is compact and T is closed, then T is upper semicontinuous.*
(c) *If T is upper semicontinuous with closed values, then T is closed.*

Definition 2.2. Let K be a nonempty subset of a topological vector space X. A multivalued map $T : K \to 2^X$ is said to be a *KKM map* provided $\mathrm{co}(A) \subseteq T(A) = \bigcup_{x \in A} T(x)$ for each finite subset A of K.

We shall use the following Fan-KKM theorem (see [8]).

Theorem 2.1 (*Fan [8]*). *Let K be a nonempty subset of a Hausdorff topological vector space E. Assume that $G : K \to 2^K \backslash \{\emptyset\}$ be a KKM map satisfying the following conditions*:

(i) *For each $x \in K$, $G(x)$ is closed*;
(ii) *For at least one $x \in K$, $G(x)$ is compact.*
Then $\bigcap_{x\in K} G(x) \neq \emptyset$.

Recall the following convexity concepts for multivalued maps.

Definition 2.3. Let K be a nonempty convex subset of a topological vector space E, Z a topological vector space and D a closed convex cone in Z. A multivalued map $F : K \to 2^Z$ is said to be *D-convex* if for every $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,

$$F(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda F(x_1) + (1-\lambda)F(x_2) - D.$$

Definition 2.4 (*Ansari and Yao [3]*). Let E and Z be topological vector spaces, X a nonempty convex subset of E, and $P : X \to 2^Z$ a multivalued map with closed convex cone values. The multivalued map $f : X \times X \to 2^Z$ is called *P_x-quasiconvex-like* if for all $x, y_1, y_2 \in X$ and $\alpha \in [0, 1]$, either

$$f(x, \alpha y_1 + (1-\alpha)y_2) \subseteq f(x, y_1) - P(x)$$

or

$$f(x, \alpha y_1 + (1-\alpha)y_2) \subseteq f(x, y_2) - P(x).$$

We need the following result on the existence of maximal elements which is a special case of a result of Deguire et al. [7].

Theorem 2.2. *Let E be a topological vector space and C be a nonempty convex subset of E. Let $S, T : C \to 2^C$ be multivalued maps satisfying the following conditions*:

(i) *For each $u \in C$, $\mathrm{co}S(u) \subseteq T(u)$.*
(ii) *For each $u \in C$, $u \notin T(u)$.*
(iii) *The set $S^{-1}(v) = \{u \in C : v \in S(u)\}$ is open for all $v \in C$.*
(iv) *If C is not compact, then there exist a nonempty compact convex subset D of C and a nonempty compact subset K of C such that for each $u \in C\backslash K$, there exists $\tilde{v} \in D$ such that $u \in S^{-1}(\tilde{v})$.*
Then there exists an $\bar{u} \in C$ such that $S(\bar{u}) = \emptyset$.

We shall use the following theorem to prove the existence of a solution of (MGIVIP).

Theorem 2.3 (*Lin et al. [16]*). *Let X be a real Hausdorff topological vector space and C be a nonempty convex subset of X. Let $f, g : C \times C \to \mathbb{R}$ be functions satisfying the following conditions*:

(i) *For all $u \in C$, $g(u, u) \geqslant 0$.*

(ii) *For each fixed* $v \in C$, *the function* $u \mapsto f(u, v)$ *is upper semicontinuous*.
(iii) *For all* $u, v \in C$, $f(u, v) < 0$ *implies* $g(u, v) < 0$.
(iv) *For each* $u \in C$, *the set* $\{v \in C : g(u, v) < 0\}$ *is convex*.
(v) *There exist a nonempty compact set* $K \subseteq C$ *and a compact convex set* $D \subseteq C$ *such that for every* $u \in C \backslash K$, *there exists* $\tilde{v} \in D$ *with* $f(u, \tilde{v}) < 0$.

Then there exists an $\bar{u} \in K$ *such that* $f(\bar{u}, v) \geqslant 0$ *for all* $v \in C$.

3. Existence of solutions of (SGIVIP)

Throughout the paper, unless otherwise specified, X and Y are assumed to be Hausdorff topological vector spaces.

We recall the following definition of segment-dense set which was introduced by Luc [17].

Definition 3.1 (*Luc [17]*). Let K be a convex set in X and K_0 a subset of K. We say that K_0 is *segment-dense in* K if for each $x \in K$ there can be found $x_0 \in K_0$ such that x is a cluster point of the set $[x, x_0] \cap K_0$, where $[x, x_0]$ denotes the line segment joining x and x_0 along with end points.

It is clear that every segment-dense subset K_0 in K is also dense but the converse statement need not be true. Luc [17] provided an example for a dense set which is not segment-dense.

Here we give another simple example of a dense set which is not segment-dense.

Example 3.1. Let K be the two dimensional Euclidean space $\mathbb{R}^2$ and define K_0 to be the set

$$K_0 := \{(p, q) \in \mathbb{R}^2 : p \in \mathbb{Q},\ \ q \in \mathbb{Q}\},$$

where $\mathbb{Q}$ denotes the set of all rational numbers. Then, it is clear that K_0 is dense in $\mathbb{R}^2$. We show that it is not segment-dense in $\mathbb{R}^2$.

Let $x=(s, t) \in \mathbb{R}^2$ such that both s and t are irrational numbers. Then for every $y=(p, q) \in K_0$, the intersection of the line segment $[x, y]$ with K_0, $[x, y] \cap K_0 = \{y\}$. So x cannot be a cluster point of the set $[x, y] \cap K_0$. Hence K_0 is not segment-dense in $\mathbb{R}^2$.

Definition 3.2. Let C be a subset of Y and C_0 be a segment-dense set in C. Let $T : C \times C \to 2^X$ and $F : X \times C \times C \to 2^{\mathbb{R}}$ be multivalued maps with nonempty values. T is said to be

(a) *pseudomonotone in the first argument with respect to* F if for all $u, v \in C$,

$$\sup F(T(u, u), v, u) \geqslant 0 \qquad \text{implies} \qquad \sup F(T(v, u), v, u) \geqslant 0; \tag{3.1}$$

(b) *weakly pseudomonotone in the first argument with respect to* F if for all $u, v \in C$,

$$\sup F(T(u, u), v, u) \geqslant 0 \qquad \text{implies} \qquad \inf F(T(v, u), v, u) \geqslant 0; \tag{3.2}$$

(c) *densely pseudomonotone* (respectively, *weakly dense pseudomonotone*) *in the first argument with respect to* F if for all $u \in C$ and $v \in C_0$, (3.1) (respectively, (3.2)) holds.

Now we prove the existence of a solution of (SGIVIP) under dense pseudomonotonicity assumption.

Theorem 3.1. *Let C be a convex subset of Y and C_0 be a segment-dense set in C. Let $T : C \times C \to 2^X$ and $F : X \times C \times C \to 2^{\mathbb{R}}$ be multivalued maps with nonempty compact values such that F is upper semicontinuous, $F(x, u, u) = \{0\}$ for all $x \in X$ and $u \in C$ and for each fixed $(x, u) \in X \times C$, the multivalued map $F(x, \cdot, u)$ is $\mathbb{R}_+$-convex. Assume that the following conditions hold*:

(i) *T is densely pseudomonotone in the first argument with respect to F.*
(ii) *For each fixed $u \in C_0$, the multivalued map $T(u, \cdot) : C \to 2^X$ is upper semicontinuous and for each fixed $v \in C$, $T(\cdot, v) : C \to 2^X$ is upper semicontinuous from the line segments in C to X.*
(iii) *There exist a nonempty compact subset $K \subseteq C$ and $\tilde{u} \in C$ such that*

$$\sup F(T(u, u), \tilde{u}, u) < 0 \quad \textit{for all } u \in C \backslash K.$$

Then (*SGIVIP*) *has a solution.*

Proof. We divide the proof into seven parts.
(a) We first consider the following problems:

$$(\mathrm{MGIVIP})^0 \quad \begin{cases} \text{Find } \bar{u} \in K \text{ such that} \\ \sup F(T(u, \bar{u}), u, \bar{u}) \geqslant 0 \quad \text{for all } u \in C_0 \end{cases}$$

and

$$(\mathrm{MGIVIP})' \begin{cases} \text{Find } \bar{u} \in K \text{ such that} \\ \sup F(T(u, \bar{u}), u, \bar{u}) \geqslant 0 \quad \text{for all } u \in C. \end{cases}$$

Observe the only difference between $(\mathrm{MGIVIP})^0$ and $(\mathrm{MGIVIP})'$ is that the set C_0 is replaced by C. We show that $(\mathrm{MGIVIP})^0$ and $(\mathrm{MGIVIP})'$ are equivalent.

Clearly, $(\mathrm{MGIVIP})'$ implies $(\mathrm{MGIVIP})^0$.

To show the reverse implication, let $\bar{u} \in K$ be a solution of $(\mathrm{MGIVIP})^0$. Let $u \in C$ be arbitrary. Since C_0 is a segment-dense set, there exists $v \in C_0$ such that u is a cluster point of $[u, v] \cap C_0$, that is, there exists a net $\{u_\alpha\}$ in $[u, v] \cap C_0$ converging to u. On the other hand, the upper semicontinuity and compactness of F and T imply the upper semicontinuity on line segment in C of the function $w \mapsto \sup F(T(w, \bar{u}), w, \bar{u})$ to X. Thus,

$$\sup F(T(u, \bar{u}), u, \bar{u}) \geqslant \overline{\lim_{\alpha}} \sup F(T(u_\alpha, \bar{u}), u_\alpha, \bar{u}) \geqslant 0.$$

This shows that $\bar{u}$ is a solution of $(\mathrm{MGVIP})'$.

(b) Now we show that $(\mathrm{MGIVIP})'$ is equivalent to (SGIVIP).

Suppose that $\bar{u} \in C$ is a solution of (SGIVIP). Then by (iii) we have $\bar{u} \in K$ and (i) implies that $\bar{u}$ is a solution of (MGIVIP)^0. From part (a), $\bar{u}$ is a solution of (MGIVIP)$'$.

For the converse, let $\bar{u} \in K$ be a solution of (MGIVIP)$'$. For fixed arbitrary $u \in C$, let $u_t := tu + (1-t)\bar{u} \in C$ for each $t \in [0, 1]$. Then we have

$$\sup F(T(u_t, \bar{u}), u_t, \bar{u}) \geqslant 0 \quad \text{for all } t \in [0, 1].$$

For fixed arbitrary $t \in (0, 1)$, we shall show that $\sup F(T(u_t, \bar{u}), u, \bar{u}) \geqslant 0$.

Since for every $(x, u) \in X \times C$, the multivalued map $v \mapsto F(x, v, u)$ is $\mathbb{R}_+$-convex and $F(x, u, u) = \{0\}$, we have

$$F(x, u_t, \bar{u}) \subseteq tF(x, u, \bar{u}) + (1-t)F(x, \bar{u}, \bar{u}) - \mathbb{R}_+ = tF(x, u, \bar{u}) - \mathbb{R}_+.$$

From this relation it follows that for every $\varepsilon > 0$ there exist $v \in F(T(u_t, \bar{u}), u_t, \bar{u})$, $w \in F(T(u_t, \bar{u}), u, \bar{u})$ and $a \geqslant 0$ such that

$$-\varepsilon \leqslant \sup\ F(T(u_t, \bar{u}), u_t, \bar{u}) - \varepsilon < v = tw - a \leqslant t \sup\ F(T(u_t, \bar{u}), u, \bar{u}),$$

and therefore,

$$\sup\ F(T(u_t, \bar{u}), u, \bar{u}) \geqslant -\frac{1}{t}\,\varepsilon.$$

Since ε is any positive number, we obtain that $\sup F(T(u_t, \bar{u}), u, \bar{u}) \geqslant 0$. Also, since $t \in (0, 1)$ was arbitrarily fixed, it follows that $\sup F(T(u_t, \bar{u}), u, \bar{u}) \geqslant 0$ for every $t \in (0, 1)$.

Let $H : [0, 1] \to \mathbb{R}$ be defined by $H(t) := \sup F(T(u_t, \bar{u}), u, \bar{u})$ for every $t \in [0, 1]$. By Lemma 2.1 (b) we may conclude that H is upper semicontinuous on $[0, 1]$ and hence $H(0) \geqslant \overline{\lim}_{t \to 0} H(t) \geqslant 0$. Thus $\sup F(T(\bar{u}, \bar{u}), u, \bar{u}) \geqslant 0$, that is, $\bar{u}$ is a solution of (SGIVIP).

In view of step (b), it is sufficient to show that (MGIVIP)$'$ has a solution.

(c) For each $w \in C$, define the multivalued maps $T_1, T_2 : C \to 2^C$ by

$$T_1(w) := \{v \in C : \sup F(T(v, v), w, v) \geqslant 0\}$$

and

$$T_2(w) := \{v \in C : \sup F(T(w, v), w, v) \geqslant 0\}.$$

It is clear that $\bigcap_{w \in C} T_1(w)$ is the set of solutions of (SGIVIP) while $\bigcap_{w \in C} T_2(w)$ is the set of solutions of (MGIVIP)$'$.

We show that T_1 is a KKM map on C. Suppose on the contrary that T_1 is not a KKM map on C. Then there exist $w_1, w_2, \ldots, w_n \in C$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \geqslant 0$ with $\sum_{j=1}^n \lambda_j = 1$ such that $\bar{w} := \sum_{j=1}^n \lambda_j w_j \notin T_1(w_i)$ for each $i \in \{1, \ldots, n\}$. Therefore,

$$\sup\ F(T(\bar{w}, \bar{w}), w_i, \bar{w}) < 0 \quad \textit{for all } i \in \{1, \ldots, n\}.$$

Since for each fixed $x \in T(\bar{w}, \bar{w}) \subseteq X$, $F(x, \cdot, \bar{w})$ is $\mathbb{R}_+$-convex, we have

$$\{0\} = F(x, \bar{w}, \bar{w}) \subseteq \sum_{j=1}^n \lambda_j F(x, w_j, \bar{w}) - \mathbb{R}_+.$$

Then there exist $u_j \in F(x, w_j, \bar{w})$ and $a \in \mathbb{R}_+$ such that

$$0 = \sum_{j=1}^{n} \lambda_j u_j - a \leqslant \sum_{j=1}^{n} \lambda_j \sup F(T(\bar{w}, \bar{w}), w_j, \bar{w}) < 0,$$

which is a contradiction. Therefore, T_1 is a KKM map on C.

(d) $T_1(\tilde{u}) \subseteq K$. Indeed if $u \in T_1(\tilde{u}) \backslash K$, then

$$\sup F(T(u, u), \tilde{u}, u) \geqslant 0$$

which contradicts (iii).

Since K is compact, $\overline{T_1(\tilde{u})}$ is also compact. Moreover,

$$\mathrm{co}\{w_1, w_2, \ldots, w_n\} \subseteq \bigcup_{i=1}^{n} T_1(w_i) \subseteq \bigcup_{i=1}^{n} \overline{T_1(w_i)}$$

for each $w_1, w_2, \ldots, w_n \in C$. By Theorem 2.1, we obtain

$$\bigcap_{w \in C} \overline{T_1(w)} \neq \emptyset. \tag{3.3}$$

(e) Next we show that $\bigcap_{w \in C_0} \overline{T_1(w)} \subseteq \bigcap_{w \in C_0} T_2(w)$.

Indeed, by step (d) $T_1(\tilde{u}) \subseteq K$ and therefore

$$\bigcap_{w \in C_0} \overline{T_1(w)} \subseteq K.$$

Let $v \in \bigcap_{w \in C_0} \overline{T_1(w)}$, then $v \in K \cap \overline{T_1(w)}$ for every $w \in C_0$. Choose an arbitrary element $u \in C_0$. We have to prove that $v \in T_2(u)$. Since $v \in \overline{T_1(u)}$, there exists a net $\{v_\alpha\}_{\alpha \in \Lambda} \subseteq T_1(u)$ such that $\{v_\alpha\}$ converges to v and therefore

$$\sup F(T(v_\alpha, v_\alpha), u, v_\alpha) \geqslant 0 \quad \text{for all } \alpha \in \Lambda.$$

By (i)

$$\sup F(T(u, v_\alpha), u, v_\alpha) \geqslant 0 \quad \text{for all } \alpha \in \Lambda.$$

Since F is upper semicontinuous with compact values, $T(u, \cdot)$ is upper semicontinuous and $T(u, v')$ is compact for all $u, v' \in C$, by Lemma 2.1 (b), for each fixed $u \in C_0$, the function $\sup F(T(u, \cdot), u, \cdot)$ is upper semicontinuous. Hence

$$\sup F(T(u, v), u, v) \geqslant \overline{\lim_{v_\alpha \to v}} \sup F(T(u, v_\alpha), u, v_\alpha) \geqslant 0.$$

Since $v \in K$ and $\sup F(T(u, v), u, v) \geqslant 0$, we obtain that $v \in T_2(u)$.

(f) From the above steps, we have

$$\bigcap_{w \in C} \overline{T_1(w)} \subseteq \bigcap_{w \in C_0} \overline{T_1(w)} \subseteq \bigcap_{w \in C_0} T_2(w) = \bigcap_{w \in C} T_2(w). \tag{3.4}$$

The relation (3.3) together with (3.4) imply that $\bigcap_{w \in C} T_2(w) \neq \emptyset$, that is, (MGIVIP)$'$ admits a solution. □

Definition 3.3. Let C be a subset of Y and C_0 be a segment-dense set in C. Let $f : X \times C \times C \to \mathbb{R}$ be a map. A multivalued map $T : C \times C \to 2^X$ is said to be

(a) *pseudomonotone in the first argument with respect to* f if for all $u, v \in C$,

$$\sup f(T(u,u), v, u) \geqslant 0 \qquad \text{implies} \qquad \sup f(T(v,u), v, u) \geqslant 0; \tag{3.5}$$

(b) *weakly pseudomonotone in the first argument with respect to* f if for all $u, v \in C$,

$$\sup f(T(u,u), v, u) \geqslant 0 \qquad \text{implies} \qquad \inf f(T(v,u), v, u) \geqslant 0; \tag{3.6}$$

(c) *densely pseudomonotone* (respectively, *weakly dense pseudomonotone*) *in the first argument with respect to* f if for all $u \in C$ and $v \in C_0$, (3.5) (respectively, (3.6)) holds.

If F is a real single-valued function, then we obtain the following result.

Corollary 3.1. *Let C be a convex subset of Y and C_0 be a segment-dense set in C. Let $T : C \times C \to 2^X$ be multivalued map with nonempty compact values and $f : X \times C \times C \to \mathbb{R}$ be a continuous function such that $f(x,u,u) = 0$ for all $x \in X$ and $u \in C$ and for each fixed $(x,u) \in X \times C$, the map $v \mapsto f(x,v,u)$ is convex. Assume that the following conditions hold*:

(i) *T is densely pseudomonotone in the first argument with respect to f.*
(ii) *For each $u \in C_0$, the multivalued map $T(u,\cdot) : C \to 2^X$ is upper semicontinuous and for each $v \in C$, $T(\cdot, v) : C \to 2^X$ is upper semicontinuous from the line segments in C to X.*
(iii) *There exist a nonempty compact subset $K \subseteq C$ and $\tilde{u} \in C$ such that*

$$\sup F(T(u,u), \tilde{u}, u) < 0 \quad \textit{for all } u \in C \setminus K.$$

Then there exists a solution $\bar{u} \in C$ of (*SIVIP*). *If in addition, $T(\bar{u}, \bar{u})$ is a convex set and for each $(u', u) \in C \times C$, the map $x \mapsto f(x, u', u)$ is concave, then there exists $\bar{x} \in T(\bar{u}, \bar{u})$ such that $f(\bar{x}, u, \bar{u}) \geqslant 0$ for all $u \in C$.*

Proof. Let $F(x,u,v) = \{f(x,u,v)\}$. Since f is continuous, F is upper semicontinuous. By Theorem 3.1, there exists $\bar{u} \in C$ such that

$$\sup f(T(\bar{u}, \bar{u}), u, \bar{u}) \geqslant 0 \quad \text{for all } u \in C$$

which means that

$$\inf_{u \in C} \sup_{x \in T(\bar{u}, \bar{u})} f(x, u, \bar{u}) \geqslant 0.$$

Now supposing the additional assumptions, it follows from Kneser's minimax theorem [13] that

$$\sup_{x \in T(\bar{u}, \bar{u})} \inf_{u \in C} f(x, u, \bar{u}) \geqslant 0.$$

Since $x \mapsto \inf_{u\in C} f(x,u,\bar{u})$ is upper semicontinuous and $T(\bar{u},\bar{u})$ is compact, there exists an element $\bar{x} \in T(\bar{u},\bar{u})$ such that

$$\inf_{u\in C} f(\bar{x},u,\bar{u}) = \max_{x\in T(\bar{u},\bar{u})} \inf_{u\in C} f(x,u,\bar{u}) \geqslant 0.$$

Hence $f(\bar{x},u,\bar{u})\geqslant 0$ for all $u \in C$. $\square$

If T is a one variable map, that is, $T : C \to 2^X$, then we derive the following result from Corollary 3.1.

Corollary 3.2. *Let C be a nonempty convex subset of Y and C_0 be a segment-dense set in C. Let $T : C \to 2^X$ be an upper semicontinuous multivalued map with nonempty compact values and $f : X \times C \times C \to \mathbb{R}$ be a continuous function such that $f(x,u,u) = 0$ for all $x \in X$ and $u \in C$, for each fixed $(x,u) \in X \times C$, the map $v \mapsto f(x,v,u)$ is convex and for each $(u',u) \in C \times C$, the map $x \mapsto f(x,u',u)$ is concave. Assume that the following conditions hold*:

(i) *T is densely pseudomonotone with respect to f.*
(ii) *There exist a nonempty compact subset $K \subseteq C$ and $\tilde{u} \in C$ such that*

$$\sup F(T(u),\tilde{u},u) < 0 \quad \textit{for all } u \in C\backslash K.$$

Then there exist $\bar{u} \in K$ and $\bar{x} \in T(\bar{u})$ such that $f(\bar{x},u,\bar{u})\geqslant 0$ for all $u \in C$.

Definition 3.4. Let C be a subset of the dual space X^* of X and C_0 be a segment-dense set in C. A multivalued map $T : C \times C \to 2^X$ is said to be

(a) *pseudomonotone in the first argument* if for all $u, v \in C$,

$$\sup\langle T(u,u), v-u\rangle \geqslant 0 \qquad \text{implies} \qquad \sup\langle T(v,u), v-u\rangle \geqslant 0; \tag{3.7}$$

(b) *weakly pseudomonotone in the first argument* if for all $u, v \in C$,

$$\sup\langle T(u,u), v-u\rangle \geqslant 0 \qquad \text{implies} \qquad \inf\langle T(v,u), v-u\rangle \geqslant 0; \tag{3.8}$$

(c) *densely pseudomonotone* (respectively, *weakly dense pseudomonotone*) *in the first argument* if for all $u \in C$ and $v \in C_0$, (3.7) (respectively, (3.8)) holds.

The following result generalizes the main results of Kassay et al. [10] and Luc [17].

Corollary 3.3. *Let X be a real Banach space with its dual space X^*. Let $C \subseteq X^*$ be a convex subset of X^* and C_0 be a segment-dense set in C. Let $T : C \times C \to 2^X$ be a multivalued map with nonempty compact values such that for each $u \in C_0$, the multivalued map $T(u,\cdot) : C \to 2^X$ is upper semicontinuous and for each $v \in C$, $T(\cdot,v) : C \to 2^X$ is upper semicontinuous from the line segments in C to X. Assume that the following assumptions hold*:

(i) *T is densely pseudomonotone in the first argument.*

(ii) *There exist a nonempty compact subset* $K \subseteq C$ *and* $\tilde{u} \in C$ *such that*

$$\sup\langle T(u,u), \tilde{u} - u\rangle < 0 \quad \textit{for all } u \in C \backslash K.$$

Then there exists $\bar{u} \in K$ *such that* $\sup\langle T(\bar{u},\bar{u}), u - \bar{u}\rangle \geqslant 0$ *for all* $u \in C$. *If in addition* $T(\bar{u},\bar{u})$ *is a convex set, then there exists* $\bar{x} \in T(\bar{u},\bar{u})$ *such that* $\langle \bar{x}, u - \bar{u}\rangle \geqslant 0$ *for all* $u \in C$.

Proof. The conclusion follows from Corollary 3.1 by letting $Y = X^*$ and $f(x, v, u) = \langle x, v - u\rangle$ for all $(x, v, u) \in X \times C \times C$. □

Remark 3.1. If $C_0 = C$, then Corollary 3.3 reduces to Theorem 3.1 of Kassay et al. [10].

The following result can be easily derived from Corollary 3.2.

Corollary 3.4 (*Theorem 4.3 [17]*). *Let* X^* *be the topological dual of* X. *Let* C *be a compact convex subset of* X *and* C_0 *be a segment-dense set in* C. *Let* $f : C \to X^*$ *be hemicontinuous* (*that is, its restrictions to line segments of* C *are continuous with respect to the weak** *topology of* X^*) *and densely pseudomonotone. Then there exists* $\bar{u} \in C$ *such that*

$$\langle f(\bar{u}), u - \bar{u}\rangle \geqslant 0 \quad \textit{for all } u \in C.$$

Now we provide an example for a mapping T verifying conditions of Corollary 3.3 on C_0 and not on C.

Example 3.2. Let X be $\mathbb{R}^2$ with the Euclidean inner product, and let

$$C := \mathrm{co}\{(0,0), (1,0), (0,-1)\}.$$

Let C_0 be the set $C\backslash\{(x_1, 0) : 0 \leqslant x_1 \leqslant 1\}$ (We take off the upper side of the triangle). Clearly, C_0 is segment-dense in C. Define $T : C \times C \to 2^{\mathbb{R}^2}$ by

$$T(u, v) := \{V(u)\}$$

for each $u, v \in C$ (T is a single-valued mapping not depending on v), where the operator $V : C \to \mathbb{R}^2$ is defined as follows:

(i) If $u = (u_1, 0) \in C\backslash C_0$, let

$$V(u) = \left(-\sin\frac{u_1\pi}{4}, \cos\frac{u_1\pi}{4}\right)$$

(rotation of the vector $(0, 1)$ with an angle equal to $u_1\pi/4$).

(ii) If $u = (u_1, u_2) \in C$ with $u_2 < 0$ (that is, $u \in C_0$), then we shall construct $V(u)$ by making use of the following property:

There exists a unique vector $w = (w_1, 0) \in C\backslash C_0$ with $u_1 < w_1 \leqslant 1$ such that the vector $u - w$ is orthogonal to $V(w)$ ($V(w)$ has been defined at step (i)). Indeed, denote by $t = (t_1, 0)$ an arbitrary vector belonging to $C\backslash C_0$ and let $\varphi(t)$ be the cosine of the angle between the vectors $u - t$ and $V(t)$ (defined also at step (i)). Then $\varphi((u_1, 0)) < 0$, $\varphi((1, 0)) \geqslant 0$ and φ

is continuous and strictly increasing on the line segment joining $(u_1, 0)$ and $(1, 0)$. Thus there exists a unique w_1 with $u_1 < w_1 \leqslant 1$ such that $\varphi((w_1, 0)) = 0$, that is, denoting by w the vector $(w_1, 0)$ we have that $u - w$ and $V(w)$ are orthogonal.

Now define $V(u) := V(w) = V((w_1, 0))$, where w is the unique vector attached to u with the above procedure. In this way, clearly V is well-defined and continuous on C. Observe also that the triangle C has been decomposed into infinitely many line segments on which our function V is constant and these line segments (level lines) are disjoint (each two lines have empty intersection). This fact allows us to show for every $u' \in C_0$ and $u \in C$ we have

$$\langle V(u), u' - u\rangle \geqslant 0 \quad \Rightarrow \quad \langle V(u'), u' - u\rangle \geqslant 0. \tag{3.9}$$

Indeed, if $u=(0, 0)$ then there is no $u' \in C_0$ such that $\langle V(u), u'-u\rangle \geqslant 0$ (observe $V((0, 0))= (0, 1)!$), consequently relation (3.9) is automatically satisfied. If $u \neq (0, 0)$ then consider the level line of V corresponding to u. It is easy to see that those vectors $u' \in C_0$ which satisfy $\langle V(u), u' - u\rangle \geqslant 0$ are situated "above" the level line of u (including the line itself). Now since the level line corresponding to u' will be above the level line of u (they do not intersect each-other) one can see that $\langle V(u'), u' - u\rangle \geqslant 0$ holds as well. Therefore, (3.9) holds.

Finally, let $u' = (1, 0) \in C$ and $u = (0, 0) \in C$. Then we have that $\langle V(u), u' - u\rangle = 0$ while $\langle V(u'), u' - u\rangle < 0$. Hence relation (3.9) fails with C instead of C_0.

4. Existence of solutions (MGIQVIP)

In this section, we establish existence results for a solution of (MGIQVIP).

Theorem 4.1. *Let C be a nonempty convex subset of Y. Let $A, B : C \to 2^C$ be multivalued maps with nonempty values such that for each $v \in C$, $A^{-1}(v)$ is open in C, $\mathrm{co}A(u) \subseteq B(u)$ for all $u \in C$ and the set $\mathscr{F} = \{u \in C : u \in B(u)\}$ is closed in C. Let $T : C \times C \to 2^X$ and $F : X \times C \times C \to 2^{\mathbb{R}}$ be lower semicontinuous multivalued maps with nonempty values such that $0 \in F(x, u, u)$ for all $(x, u) \in X \times C$. Assume that the following conditions hold*:

(i) *T is weakly pseudomonotone in the first argument with respect to F.*
(ii) *For each $u \in C$, the set*

$$Q(u) = \{v \in C : \sup F(T(u, u), v, u) < 0\} \textit{ is convex}.$$

(iii) *There exist a nonempty compact set $K \subseteq C$ and a nonempty compact convex set $D \subseteq C$ such that for every $u \in C\backslash K$, there exists $\tilde{v} \in D$ with $\tilde{v} \in A(u)$ such that $\inf F(T(\tilde{v}, u), \tilde{v}, u) < 0$.*

Then (MGIQVIP) has a solution.

Proof. Let $P : C \to 2^C$ be defined by

$$P(u) = \{v \in C : \inf F(T(v, u), v, u) < 0\} \quad \text{for all } u \in C.$$

Define two multivalued maps $S, T' : C \to 2^C$ by

$$T'(u) = \begin{cases} B(u) \cap Q(u) & \text{if } u \in \mathscr{F}, \\ B(u) & \text{if } u \in C \backslash \mathscr{F} \end{cases}$$

and

$$S(u) = \begin{cases} A(u) \cap P(u) & \text{if } u \in \mathscr{F}, \\ A(u) & \text{if } u \in C \backslash \mathscr{F}. \end{cases}$$

By weak pseudomonotonicity of T we have, $P(u) \subseteq Q(u)$ for all $u \in C$. Since $Q(u)$ is convex, we have $\mathrm{co}P(u) \subseteq Q(u)$ for all $u \in C$ and therefore $\mathrm{co}S(u) \subseteq T'(u)$. Since $0 \in F(x, u, u)$ for all $(x, u) \in X \times C$, we have

$$\sup\ F(T(u, u), u, u) \geqslant 0 \quad \text{for all } u \in C.$$

Therefore $u \notin Q(u)$ and so $u \notin T'(u)$ for all $u \in C$. By using lower semicontinuity of F and T and Lemma 2.1 (a) we have, for each fixed $v \in C$, $u \mapsto \sup[-F(T(v, u), v, u)]$ is lower semicontinuous. Thus for each $v \in C$,

$$u \mapsto \inf\ F(T(v, u), v, u) = -\sup[-F(T(v, u), v, u)] \text{ is upper semicontinuous.}$$

Hence, for each $v \in C$,

$$P^{-1}(v) = \{u \in C : \inf F(T(v, u), v, u) < 0\}$$

is open in C. Since $\mathscr{F}$ is closed in C and for each $v \in C$, $A^{-1}(v)$ is open in C, it is easy to see that

$$S^{-1}(v) = (A^{-1}(v) \cap P^{-1}(v)) \cup (A^{-1}(v) \cap (C \backslash \mathscr{F}))$$

is open in C. By (iii), there exist a nonempty compact set $K \subseteq C$ and a nonempty compact convex set $D \subseteq C$ such that for every $u \in C \backslash K$, there exists $\tilde{v} \in D$ with $\tilde{v} \in A(u)$ such that $\inf F(T(\tilde{v}, u), \tilde{v}, u) < 0$. For such u and $\tilde{v}$ we have $u \in A^{-1}(\tilde{v}) \cap P^{-1}(\tilde{v})$. Thus $u \in S^{-1}(\tilde{v})$. Hence all the conditions of Theorem 2.2 are satisfied, therefore there exists $\bar{u} \in C$ such that $S(\bar{u}) = \emptyset$.

If $\bar{u} \in C \backslash \mathscr{F}$, then $A(\bar{u}) = S(\bar{u}) = \emptyset$ which contradicts with $A(u) \neq \emptyset$ for all $u \in C$. Therefore $\bar{u} \in \mathscr{F}$. Hence $\bar{u} \in B(\bar{u})$ and $A(\bar{u}) \cap P(\bar{u}) = \emptyset$. Then, for all $v \in A(\bar{u})$, $v \notin P(\bar{u})$. That is, $\bar{u} \in B(\bar{u})$ and

$$\inf\ F(T(v, \bar{u}), v, \bar{u}) \geqslant 0 \quad \text{for all } v \in A(\bar{u}).$$

The proof is completed. □

Remark 4.1. If for each fixed $(x, u) \in X \times C$, the multivalued map $v \mapsto F(x, v, u)$ is $\mathbb{R}_+$-quasiconvex-like, then condition (ii) of Theorem 4.1 holds.

Proof. For each $u \in C$, let $v_1, v_2 \in Q(u) = \{v \in C : \sup F(T(u, u), v, u) < 0\}$ and $\lambda \in [0, 1]$. Then

$$\sup\ F(T(u, u), v_1, u) < 0 \quad \text{and} \quad \sup\ F(T(u, u), v_2, u) < 0.$$

Since for all $(x, u) \in X \times C$, $F(x, \cdot, u)$ is $\mathbb{R}_+$-quasiconvex-like, either

$$F(x, \lambda v_1 + (1-\lambda)v_2, u) \subseteq F(x, v_1, u) - \mathbb{R}_+$$

or

$$F(x, \lambda v_1 + (1-\lambda)v_2, u) \subseteq F(x, v_2, u) - \mathbb{R}_+.$$

Let us take $F(x, \lambda v_1 + (1-\lambda)v_2, u) \subseteq F(x, v_1, u) - \mathbb{R}_+$, then we have

$$\begin{aligned}\sup\, F(T(u,u), \lambda v_1 + (1-\lambda)v_2, u) &\leqslant \sup[F(T(u,u), v_1, u) - \mathbb{R}_+]\\ &= \sup\, F(T(u,u), v_1, u) + \sup(-\mathbb{R}_+)\\ &= \sup\, F(T(u,u), v_1, u) + 0 < 0.\end{aligned}$$

A similar argument leads to the same result in case $F(x, \lambda v_1+(1-\lambda)v_2, u) \subseteq F(x, v_2, u) - \mathbb{R}_+$. Therefore $\lambda v_1 + (1-\lambda)v_2 \in Q(u)$ and thus $Q(u)$ is convex. □

When F is a single-valued map, we have the following result.

Theorem 4.2. *Let C be a nonempty convex subset of Y. Let $A, B : C \to 2^C$ be multivalued maps with nonempty values such that for each $v \in C$, $A^{-1}(v)$ is open in C,* $\mathrm{co}A(u) \subseteq B(u)$ *for all $u \in C$ and the set $\mathscr{F} = \{u \in C : u \in B(u)\}$ is closed in C. Let $T : C \times C \to 2^X$ be a lower semicontinuous multivalued map with nonempty values and $f : X \times C \times C \to \mathbb{R}$ be an upper semicontinuous function such that $f(x, u, u) = 0$ for all $(x, u) \in X \times C$ and for each fixed $(x, u) \in X \times C$, $f(x, \cdot, u)$ is quasiconvex. Assume that the following conditions hold*:

(i) *T is weakly pseudomonotone in the first argument with respect to f.*
(ii) *There exist a nonempty compact set $K \subseteq C$ and a nonempty compact convex set $D \subseteq C$ such that for every $u \in C \backslash K$, there exists $\tilde{v} \in D$ with $\tilde{v} \in A(u)$ such that* $\inf\, f(T(\tilde{v}, u), \tilde{v}, u) < 0$.

Then there exists a solution $\bar{u} \in C$ of (MIQVIP)*, that is, $\bar{u} \in B(\bar{u})$ and* $\inf\, f(T(v, \bar{u}), v, \bar{u}) \geqslant 0$ *for all $v \in A(\bar{u})$.*

In particular, if f is continuous and $T(u, v)$ is compact for all $(u, v) \in C \times C$, then there exists $\bar{u} \in C$ with $\bar{u} \in B(\bar{u})$ such that for each $v \in A(\bar{u})$ there exists $\bar{y}_v \in T(v, \bar{u})$ satisfying $f(\bar{y}_v, v, \bar{u}) \geqslant 0$.

Proof. Since f is upper semicontinuous, it follows from Berge's theorem [5] that for each fixed $v \in C$,

$$u \mapsto \inf\, f(T(v, u), v, u) = -\sup[-f(T(v, u), v, u)]$$

is upper semicontinuous. Since for each $(x, u) \in X \times C$, $f(x, \cdot, u)$ is quasiconvex, by Remark 4.1, for each $u \in C$, the set $\{v \in C : \sup\, f(T(u, u), v, u) < 0\}$ is convex. Then by Theorem 2.2 and following the argument as in the proof of Theorem 4.1, there exists $\bar{u} \in C$ such that $\bar{u} \in B(\bar{u})$ satisfying

$$\inf\, f(T(v, \bar{u}), v, \bar{u}) \geqslant 0 \quad \text{for all } v \in A(\bar{u}).$$

If $T(u, v)$ is compact for all $(u, v) \in C \times C$ and f is continuous, then for each $v \in A(\bar{u})$, there exists $\bar{y}_v \in T(v, \bar{u})$ such that

$$f(\bar{y}_v, v, \bar{u}) = \min\ f(T(v, \bar{u}), v, \bar{u}) \geqslant 0.$$

This completes the proof. □

The following corollary can be easily derived from Theorem 4.2.

Corollary 4.1. *Let E be a reflexive Banach space with its dual E^* and C be a nonempty convex subset of E. Let $A, B : C \to 2^C$ be multivalued maps with nonempty values such that for each $v \in C$, $A^{-1}(v)$ is open in C, $\mathrm{co}A(u) \subseteq B(u)$ for all $u \in C$ and the set $\mathscr{F} = \{u \in C : u \in B(u)\}$ is closed in C. Let $T : C \times C \to 2^{E^*}$ be a lower semicontinuous multivalued map with nonempty values. Assume that the following conditions hold*:

(i) *T is weakly pseudomonotone in the first argument.*
(ii) *There exist a nonempty compact set $K \subseteq C$ and a nonempty compact convex set $D \subseteq C$ such that for every $u \in C \backslash K$, there exists $\tilde{v} \in D$ with $\tilde{v} \in A(u)$ satisfying* $\inf\langle T(\tilde{v}, u), \tilde{v} - u\rangle < 0$.

Then (*MGQVIP*) *has a solution.*

Proof. Let $Y = E$, $X = E^*$, and $f(x, v, u) = \langle x, v - u\rangle$ for all $(x, v, u) \in X \times C \times C$. Clearly, $f(x, u, u) = 0$ for all $(x, u) \in X \times C$, and f is continuous. Since for each $u \in C$ and $x \in X$, $v \mapsto \langle x, v - u\rangle = f(x, v, u)$ is affine, hence it is convex. Then for each $(x, u) \in X \times C$, $f(x, \cdot, u)$ is quasiconvex. The result follows from Theorem 4.2. □

Now, we prove the existence of a solution of (MGIVIP) under weak dense pseudomonotonicity assumption.

Theorem 4.3. *Let C a subset of Y and C_0 be a convex and segment-dense set in C. Let $T : C \times C \to 2^X$ and $F : X \times C \times C \to 2^{\mathbb{R}}$ be lower semicontinuous multivalued maps with nonempty values such that $0 \in F(x, u, u)$ for all $(x, u) \in X \times C_0$. Assume that the following conditions hold*:

(i) *T is weakly dense pseudomonotone in the first argument with respect to F.*
(ii) *For each $(x, u) \in X \times C_0$, $F(x, \cdot, u)$ is $\mathbb{R}_+$-quasiconvex-like.*
(iii) *There exist a nonempty compact set $K \subseteq C_0$ and a nonempty compact convex set $D \subseteq C_0$ such that for every $u \in C_0 \backslash K$, there exists $\tilde{v} \in D$ satisfying* $\inf\ F(T(\tilde{v}, u), \tilde{v}, u) < 0$.

Then there exists a solution $\bar{u} \in C$ of (*MGIVIP*), *that is,* $\inf\ F(T(v, \bar{u}), v, \bar{u}) \geqslant 0$ *for all $v \in C$.*

Proof. Let $f, g : C \times C \to \mathbb{R}$ be defined by

$$f(u, v) = \inf\ F(T(v, u), v, u) \quad \text{and} \quad g(u, v) = \sup\ F(T(u, u), v, u)$$

for all $u, v \in C$. Following the argument as in Theorem 4.1, it is easy to see that f is upper semicontinuous in each argument.

Now we shall show that the restrictions of the functions f and g to the set $C_0 \times C_0$ satisfy all the conditions of Theorem 2.3. Indeed, assumption (i) implies condition (iii) of Theorem 2.3. Since $0 \in F(x,u,u)$ for all $(x,u) \in X \times C_0$, we have $g(u,u) \geqslant 0$ for all $u \in C_0$. Furthermore, assumption (ii) and Remark 4.1 imply that for each $u \in C_0$, the set

$$\{v \in C_0 : \sup\ F(T(u,u), v, u) < 0\} \text{ is convex.}$$

It follows from Theorem 2.3 that there exists $\bar{u} \in K$ such that

$$\inf\ F(T(v,\bar{u}), v, \bar{u}) \geqslant 0 \quad \text{for all } v \in C_0.$$

It remains to show that the above inequality holds for every $v \in C$.

Let $v \in C$. Since C_0 is segment-dense in C, there exist $u_0 \in C_0$ and a net $\{v_\alpha\}$ in $[v, u_0] \cap C_0$ such that $v_\alpha \to v$. Since f is upper semicontinuous in the second argument, we conclude that

$$f(\bar{u}, v) \geqslant \overline{\lim_{v_\alpha \to v}}\, f(\bar{u}, v_\alpha) = \overline{\lim_{v_\alpha \to v}}\, \inf\ F(T(v_\alpha, \bar{u}), v_\alpha, \bar{u}) \geqslant 0.$$

Therefore $\inf F(T(v,\bar{u}), v, \bar{u}) \geqslant 0$ for all $v \in C$. $\square$

Corollary 4.2. *Let C be a subset of Y and C_0 be a convex and segment-dense set in C. Let $T : C \times C \to 2^X$ be a lower semicontinuous multivalued map with nonempty values and $f : X \times C \times C \to \mathbb{R}$ be an upper semicontinuous function such that $f(x,u,u) = 0$ for all $(x,u) \in X \times C_0$. Assume that the following conditions hold*:

(i) *T is weakly dense pseudomonotone in the first argument with respect to f.*
(ii) *For each $(x,u) \in X \times C_0$, $f(x,\cdot,u)$ is quasiconvex.*
(iii) *There exist a nonempty compact set $K \subseteq C_0$ and a nonempty compact convex set $D \subseteq C_0$ such that for every $u \in C_0 \backslash K$, there exists $\tilde{v} \in D$ satisfying* $\inf\ f(T(\tilde{v},u), \tilde{v}, u) < 0$.

Then there exists a solution $\bar{u} \in C$ of (*MIVIP*), *that is,* $\inf\ f(T(v,\bar{u}), v, \bar{u}) \geqslant 0$ *for all $v \in C$.*

In particular, if f is continuous and $T(u,v)$ is compact for all $(u,v) \in C \times C$, then there exists $\bar{u} \in C$ such that for each $v \in C$, there exists $\bar{y}_v \in T(v,\bar{u})$ such that $f(\bar{y}_v, v, \bar{u}) \geqslant 0$.

Definition 4.1. Let *X^* be the dual space of X. Let C be a subset of X and C_0 be a segment-dense set in C. A multivalued map $T : C \to 2^{X^*}$ is said to be*

(a) *weakly pseudomonotone* if for all $u, v \in C$,

$$\sup\langle T(u), v-u\rangle \geqslant 0 \qquad \text{implies} \qquad \inf\langle T(v), v-u\rangle \geqslant 0; \tag{4.1}$$

(b) *weakly dense pseudomonotone* if for all $u \in C$ and $v \in C_0$, (4.1) holds.

From Corollary 4.2 we obtain the following result which extends Theorem 5.3 of Lin et al. [16].

Corollary 4.3. *Let X be a reflexive Banach space with its dual X^*. Let C be a subset of X and C_0 be a convex and segment-dense set in C. Let $T : C \to 2^{X^*}$ be weakly* *dense pseudomonotone and lower semicontinuous from the norm topology of* X *to the weak topology of* X^*. *Assume that there exists a nonempty compact subset* K *of* C_0 *and an element* $\tilde{v} \in C_0$ *such that* $\inf\langle T(\tilde{v}), \tilde{v} - u\rangle < 0$ *for all* $u \in C_0\backslash K$. *Then there exists a solution* $\bar{u} \in C$ *of* (*MVIP*), *that is*, $\inf\langle T(u), u - \bar{u}\rangle \geqslant 0$ *for all* $u \in C$.

Remark 4.2. When $C = C_0$, Corollary 4.3 reduces to Theorem 5.3 in [16].

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