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System of Generalized Vector Quasi-Equilibrium Problems with Applications

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1. Introduction and Formulations

Let I be any index set and for each $i \in I$, let X_i be a Hausdorff topological vector space and K_i a nonempty convex subset of X_i. We set $K = \prod_{i\in I} K_i$, $X = \prod_{i\in I} X_i$ and $K^i = \prod_{j\in I,\ j\neq i} K_j$, and we write $K = K^i \times K_i$. For $x \in K$, x^i denotes the projection of x onto K^i and hence we also write $x = (x^i, x_i)$. For each $i \in I$, let Y_i be a topological vector space and let $C_i : K \to 2^{Y_i}$ be a multivalued map such that for each $x \in K$, $C_i(x)$ is a proper, closed and convex cone with int $C_i(x) \neq \emptyset$, where int C and 2^Y denote the interior of C and the family of subsets of Y, respectively. For each $i \in I$, let $F_i : K \times K_i \to 2^{Y_i}$ and $A_i : K \to 2^{K_i}$ be multivalued maps with nonempty values. We consider the following *system of generalized vector quasi-equilibrium problems:*

$$\text{(SGVQEP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I,\ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ F_i(\bar{x}, y_i) \not\subseteq -\text{int}\, C_i(\bar{x}), \quad \text{for all } \ y_i \in A_i(\bar{x}). \end{cases}$$

If for each $i \in I$ and for all $x \in K$, $A_i(x) = K_i$, then (SGVQEP) reduces to the system of generalized vector equilibrium problems (for short, SGVEP) which is introduced and studied by Ansari et al [5] with applications to Nash equilibrium problem for vector-valued functions.

If I is a singleton set, then (SGVQEP) reduces to a generalized vector quasi-equilibrium problem which contains generalized implicit vector quasi-variational inequality problem, generalized vector quasi-variational inequality and variational-like inequality problems and vector quasi- equilibrium problems as special cases. For further detail on generalized vector quasi-equilibrium problems and their applications, we refer [2] and references therein.

Examples of (SGVQEP):

For each $i \in I$, we denote by $L(X_i, Y_i)$ the space of all continuous linear operators from X_i into Y_i and let D_i be a nonempty subset of $L(X_i, Y_i)$. For each $i \in I$, let $T_i : K \to 2^{D_i}$ be a multivalued map with nonempty values.

(1) *System of Generalized Implicit Vector Quasi-Variational Inequalities*:
For each $i \in I$, let $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a vector-valued map. The problem of system of generalized implicit vector quasi-variational inequalities (for short, SGIVQVIP) is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}),\ \ \exists \bar{u}_i \in T_i(\bar{x}) \ \ : \ \ \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\text{int}\, C_i(\bar{x}).$$

Setting for each $i \in I$,

$$F_i(x, y_i) = \psi_i(T_i(x), x_i, y_i) = \{\psi_i(u_i, x_i, y_i) : u_i \in T_i(x)\}.$$

Then (SGVQEP) coincides with (SGIVQVIP).

For $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_-$ for all $x \in K$ and for each $i \in I$, (SGIVQVIP) is called the *problem of system of generalized implicit quasi-variational inequalities*. Further, for all $x \in K$ and for each $i \in I$, $A_i(x) = K_i$, it is called the *problem of system of generalized implicit variational inequalities*. Ansari and Yao [7] studied such a problem with application to Nash equilibrium problem [20].

If I is a singleton set, (SGIVQVIP) reduces to the *generalized implicit vector quasi-variational inequality problem.*

The (SGIVQVIP) contains the following problems as special cases:

(i) For each $i \in I$, let $\theta_i : K \times D_i \longrightarrow D_i$ and $\eta_i : K_i \times K_i \longrightarrow X_i$ be bifunctions. If for each $i \in I$,

$$\psi_i(T_i(x), x_i, y_i) = \langle \theta_i(x, T_i(x)), \eta_i(y_i, x_i) \rangle = \{\langle \theta_i(x, u_i), \eta_i(y_i, x_i) \rangle : u_i \in T_i(x)\},$$

then (SGIVQVIP) reduces to the *problem of system of generalized vector quasi-variational-like inequalities* (for short, SGV QVLIP) (I) which is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}), \quad \exists \bar{u}_i \in T_i(\bar{x}) \quad : \quad \langle \theta_i(\bar{x}, \bar{u}_i), \eta_i(y_i, \bar{x}_i) \rangle \notin -\text{int } C_i(\bar{x}),$$

where $\langle s_i, x_i \rangle$ denotes the evaluation of $s_i \in L(X_i, Y_i)$ at $x_i \in X_i$. If I is a singleton set, then (SGVQVLIP)(I) becomes the *generalized vector quasi-variational-like inequality problem*. The strong solution (that is, $\bar{u}_i$ does not depend on y_i) of (SGVQVLIP)(I) is studied by Chen et al [12] and Lee et al [19], see also references therein.

If for each $i \in I$, $\theta_i(x, u_i) = u_i$ for all $x \in K$, then (SGVQVLI P) (I) becomes the following problem denoted by (SGVQVLIP) (II): Find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}), \quad \exists \bar{u}_i \in T_i(\bar{x}) \quad : \quad \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i) \rangle \notin -\text{int } C_i(\bar{x}).$$

For $Y_i = \mathbb{R}$, $C_i(x) = \mathbb{R}_-$ and $A_i(x) = K_i$ for all $x \in K$ and for each $i \in I$, this problem is studied in [7] with application to the Nash equilibrium problem [20].

(ii) If for each $i \in I$,

$$\psi_i(T_i(x), x_i, y_i) = \langle T_i(x), y_i - x_i \rangle = \{\langle u_i, y_i - x_i \rangle : u_i \in T_i(x)\},$$

then (SGIVQVIP) reduces to the *problem of system of generalized vector quasi-variational inequalities* (for short, SGVQVI P) which is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}), \quad \exists \bar{u}_i \in T_i(\bar{x}) \quad : \quad \langle \bar{u}_i, y_i - \bar{x}_i \rangle \notin -\text{int } C_i(\bar{x}).$$

(2) *System of Vector Quasi-Equilibrium Problems* (for short, SVQEP): For each $i \in I$, let F_i be a single-valued map, then (SGVQEP) is equivalent to the following *system of vector quasi-equilibrium problems*:

$$\text{(SVQEP)} \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I,\ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ F_i(\bar{x}, y_i) \notin -\text{int } C_i(\bar{x}), \quad \text{for all } y_i \in A_i(\bar{x}). \end{cases}$$

It is introduced and studied by Ansari et al [1] with applications to the Debreu type equilibrium problems for differentiable vector-valued functions.

The (SVQEP) contains the following problems as special cases:

(i) If for each $i \in I$, $F_i(x, y_i) = \langle T_i(x), y_i - x_i \rangle$, where T_i is a single-valued operator and x_i is the ith component of x, then (SVQEP) reduces to the *problem of system of vector quasi-variational inequalities* (for short, SVQVIP) which is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\langle T_i(\bar{x}), y_i - \bar{x}_i \rangle \notin -\text{int } C_i(\bar{x}), \quad \text{for all } y_i \in A_i(\bar{x}).$$

If for each $i \in I$ and for all $x \in K$, $C_i(x) = \mathbb{R}_+$ and $Y_i = \mathbb{R}$, then (SVQVIP) reduces to the *system of quasi-variational inequalities* studied by Ansari et al [3].

(ii) For each $i \in I$, let $\varphi_i : K \to Y_i$ be a vector-valued function. If for each $i \in I$,

$$F_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x),$$

then (SVQEP) is equivalent to the following Debreu type equilibrium problem for vector-valued functions [13]:

$$\text{(DEP)} \begin{cases} \text{Find } \bar{x} \in K \text{ such that } \forall i \in I,\ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ \varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x}), \text{ for all } y_i \in A_i(\bar{x}). \end{cases}$$

From the above examples/special cases, it is clear that (SGVQEP) is more general and unified format which contains many known problems as special cases. A comprehensive bibliography on vector equilibrium problems and vector variational inequalities can be found in a recent volume [17] edited by F. Giannessi.

In this paper, we establish some existence results for a solution to (SGVQEP) with or without involving Φ-condensing maps. As consequences, we prove the existence of a solution of many known problems mentioned above. Ansari et al [1] used (SVQEP) as a tool to prove the existence of a solution of Debreu type equilibrium problem for vector-valued but differentiable functions. As applications of our results we derive the existence results for a solution of Debreu type equilibrium problem for vector-valued but nondifferentiable functions.

2. Preliminaries

In this section, we recall some definitions and results which will be used in the sequel.

Definition 2.1. Let W and Z be topological vector spaces. A multivalued map $T: W \to 2^Z \setminus \{\emptyset\}$ is said to be *closed* if its graph is closed in $W \times Z$.

Definition 2.2. [8] Let W and Z be topological vector spaces. A multivalued map $T: W \to 2^Z \setminus \{\emptyset\}$ is called *upper semi-continuous on* W if T has compact values and for each $x_0 \in W$ and for any open set V in Z containing $T(x_0)$ there exists an open neighborhood U of x_0 in W such that $T(x) \subseteq V$ for all $x \in U$.

Definition 2.3. [21, 22] Let E be a Hausdorff topological vector space and L a lattice with least element, denoted by $\mathbf{0}$. A mapping $\Phi : 2^E \to L$ is called a *measure of noncompactness* provided that the following conditions hold for any $M, N \in 2^E$:

(i) $\Phi(M) = \mathbf{0}$ if and only if M is precompact (i.e., it is relatively compact).
(ii) $\Phi(\overline{\text{conv}}M) = \Phi(M)$, where $\overline{\text{conv}}M$ denotes the closed convex hull of M.
(iii) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

It follows from (iii) that if $M \subseteq N$, then $\Phi(M) \leq \Phi(N)$.

Definition 2.4. [21, 22] Let $\Phi : 2^E \to L$ be a measure of noncompactness on E and $D \subseteq E$. A multivalued map $T: D \to 2^E$ is called Φ-*condensing* provided that if $M \subseteq D$ with $\Phi(T(M)) \geq \Phi(M)$ then M is relatively compact.

Remark 2.1. Note that every multivalued map defined on a compact set is necessarily Φ-condensing. If E is locally convex, then a compact multivalued map (i.e., $T(D)$ is precompact) is Φ-condensing for any measure of noncompactness Φ. Obviously, if $T : D \to 2^E$ is Φ-condensing and if $T' : D \to 2^E$ satisfies $T'(x) \subseteq T(x)$ for all $x \in D$, then T' is also Φ-condensing.

Definition 2.5. Let W and Z be topological vector spaces. A point $\bar{x} \in W$ is said to be a *maximal element* of a multivalued maps $T : W \to 2^Z$ if $T(\bar{x}) = \emptyset$.

We shall use the following maximal element results for a family of multivalued maps to establish the existence results for a solution of (SGVQEP).

Theorem 2.1. [14] Let I be any index set. For each $i \in I$, let K_i be a nonempty and convex subset of a Hausdorff topological vector space X_i, and let $S_i : K = \prod_{i \in I} K_i \to 2^{K_i}$ be a multivalued map. Assume that the following conditions hold:

(i) For each $i \in I$ and for all $x \in K$, $S_i(x)$ is convex.
(ii) For each $i \in I$ and for all $x \in K$, $x_i \notin S_i(x)$, where x_i is the ith component of x.
(iii) For each $i \in I$ and for all $y_i \in K_i$, $S_i^{-1}(y_i)$ is open in K.
(iv) There exist a nonempty and compact subset N of K and a nonempty, compact and convex subset B_i of K_i for each $i \in I$, such that for all $x \in K \setminus N$ there exists $i \in I$ satisfying $S_i(x) \cap B_i \neq \emptyset$.

Then there exists $\bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Remark 2.2. If for each $i \in I$, K_i is a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X_i, then condition (iv) of Theorem 2.1 can be replaced by the following condition:

(iv)′ The multivalued map $S : K \to 2^K$ defined as $S(x) := \prod_{i\in I} S_i(x)$ for all $x \in K$, is Φ-condensing.

(See, Corollary 4 in [11]).

3. Existence Results

Throughout this paper, unless otherwise specified, we assume that I is any index set and for each $i \in I$, Y_i is a topological vector space, $K = \prod_{i\in I} K_i$, $C_i : K \to 2^{Y_i}$ is a multivalued map such that for all $x \in K$, $C_i(x)$ is a proper, closed and convex cone with int $C_i(x) \neq \emptyset$. For each $i \in I$, we also assume that $A_i : K \to 2^{K_i}$ is a multivalued map such that for all $x \in K$, $A_i(x)$ is nonempty and convex, $A_i^{-1}(y_i)$ is open in K for all $y_i \in K_i$ and the set $\mathcal{F}_i := \{x \in K : x_i \in A_i(x)\}$ is closed in K, where x_i is the ith component of x.

Theorem 3.1. For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space $X_i$an d let $F_i : K \times K_i \to 2^{Y_i}$ be a multivalued map with nonempty values. For each $i \in I$, assume that

(i) for all $x \in K$, $F_i(x, x_i) \not\subseteq -\text{int } C_i(x)$, where x_i is the ith component of x;
(ii) for all $x \in K$, the set $\{y \in K_i : F_i(x, y_i) \subseteq -\text{int } C_i(x)\}$ is convex;
(iii) for all $y_i \in K_i$, the set $\{x \in K : F_i(x, y_i) \not\subseteq -\text{int } C_i(x)\}$ is closed in K;
(iv) there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ satisfying $\tilde{y}_i \in A_i(x)$ and $F_i(x, \tilde{y}_i) \subseteq -\text{int } C_i(x)$.

Then (SGVQEP) has a solution.

Proof. For each $i \in I$, we define a multivalued map $Q_i : K \to 2^{K_i}$ by

$$Q_i(x) = \{y_i \in K_i : F_i(x, y_i) \subseteq -\text{int } C_i(x)\}, \quad \text{for all } x \in K.$$

By condition (ii), for each $i \in I$ and for all $x \in K$, $Q_i(x)$ is convex. Condition (iii) implies that for each $i \in I$ and for all $y_i \in K_i$, $Q_i^{-1}(y_i) = \{x \in K : F_i(x, y_i) \subseteq -\text{int } C_i(x)\}$ is open in K. Condition (i) implies that for each $i \in I$ and for all $x \in K$, $x_i \notin Q_i(x)$.

For each $i \in I$ and for all $x \in K$, we define another multivalued map $S_i : K \to 2^{K_i}$ by

$$S_i(x) = \begin{cases} A_i(x) \cap Q_i(x), & \text{if } x \in \mathcal{F}_i \\ A_i(x), & \text{if } x \in K \setminus \mathcal{F}_i. \end{cases}$$

Then, clearly for each $i \in I$ and for all $x \in K$, $S_i(x)$ is convex and $x_i \notin S_i(x)$. Since for each $i \in I$ and for all $y_i \in K_i$,

$$S_i^{-1}(y_i) = \left(A_i^{-1}(y_i) \cap Q_i^{-1}(y_i)\right) \bigcup \left((K \setminus \mathcal{F}_i) \cap A_i^{-1}(y_i)\right)$$

(see, for example, the proof of Lemma 2.3 in [15]) and $A_i^{-1}(y_i)$, $Q_i^{-1}(y_i)$ and $K \setminus \mathcal{F}_i$ are open in K, we have $S_i^{-1}(y_i)$ is open in K.

Condition (iv) of Theorem 2.1 is followed from condition (iv). Hence by Theorem 2.1, there exists $\bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$. Since for each $i \in I$ and for all $x \in K$, $A_i(x)$ is nonempty, we have $A_i(\bar{x}) \cap Q_i(\bar{x}) = \emptyset$ for each $i \in I$. Therefore for each $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) \text{ and } F_i(\bar{x}, y_i) \not\subseteq -\text{int } C_i(\bar{x}), \quad \text{for all } y_i \in A_i(\bar{x}). \qquad \square$$

Next we establish an existence result for a solution to (SGVQEP) involving Φ-condensing maps.

Theorem 3.2. For each $i \in I$, let K_i be a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X_i, $F_i : K \times K_i \to 2^{Y_i}$ a multivalued map with nonempty values and let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Assume that the conditions (i) - (iii) of Theorem 3.1 hold. Then (SGVQEP) has a solution.

Proof. In view of Remark 2.2, it is sufficient to show that the multivalued map $S : K \to 2^K$ defined as $S(x) = \prod_{i \in I} S_i(x)$ for all $x \in K$, is ϕ-condensing, where S_i's are the same as defined in the proof of Theorem 3.1. By the definition of S_i, $S_i(x) \subseteq A_i(x)$ for each $i \in I$ and for all $x \in K$ and therefore $S(x) \subseteq A(x)$ for all $x \in K$. Since A is Φ-condensing, by Remark 2.1, we have S is also Φ-condensing. $\square$

Definition 3.1. [6] Let W and Z be topological vector spaces and M a nonempty convex subset of W and let $P : M \to 2^Z$ be a multivalued map such that for each $x \in M$, $P(x)$ is a closed, convex cone with nonempty interior. A multivalued map $F : M \times M \to 2^Z \setminus \{\emptyset\}$ is called *$P(x)$-quasiconvex-like* if for all $x, y_1, y_2 \in M$ and $t \in [0,1]$, we have either

$$F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_1) - P(x),$$

or

$$F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_2) - P(x).$$

Remark 3.1. (a) If for each $i \in I$, F_i is C_x-quasiconvex-like, then the set $\{y_i \in K_i : F_i(x, y_i) \subseteq -\text{int } C_i(x)\}$ is convex, for each $x \in K$ (see, for example, the proof of Theorem 2.1 in [6]).
(b) If for each $i \in I$, X_i is locally convex Hausdorff topological vector space, the multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{-\text{int } C_i(x)\}$ for all $x \in K$, is closed on K and for each $y \in K$, $F(.,y)$ is upper semicontinuous on K, then condition (iii) of Theorem 3.1 is satisfied; See, for example, the proof of Theorem 2.1 in [6].

Definition 3.2. [4] Let W and Z be topological vector spaces, M a nonempty convex subset of W and D a nonempty subset of $L(W,Z)$. Let $T : M \to 2^D \setminus \{\emptyset\}$ and $P : M \to 2^Z$ be multivalued maps such that for each $x \in M$, $P(x)$ is a closed, convex cone with nonempty interior. A function $\psi : D \times M \times M \to Z$ is called *$P(x)$-quasiconvex-like* if for all $x, y_1, y_2 \in M$ and $t \in [0,1]$, we have either for all $u \in T(x)$,

$$\psi(u, x, ty_1 + (1-t)y_2) \in \psi(u, x, y_1) - P(x),$$

or

$$\psi(u, x, ty_1 + (1-t)y_2) \in \psi(u, x, y_2) - P(x).$$

From Theorems 3.1 and 3.2, we derive the following existence result for a solution of (SGIVQVIP).

Corollary 3.1. For each $i \in I$, let K_i be a nonempty convex subset of a locally convex topological vector space X_i and let D_i be a nonempty subset of $L(X_i, Y_i)$. For each $i \in I$, $T_i : K \to 2^{D_i}$ be an upper semicontinuous multivalued map with nonempty values and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a vector-valued map. For each $i \in I$, assume that

(i) the multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{-\text{int } C_i(x)\}$ for all $x \in K$, is closed on K;
(ii) for all $x \in K$ and $u_i \in T_i(x)$, $\psi_i(u_i, x_i, x_i) \notin -\text{int } C_i(x)$, where x_i is the ith component of x;
(iii) ψ_i is $C_i(x)$-quasiconvex-like;
(iv) for all $y_i \in K_i$, the map $(u_i, x_i) \mapsto \psi_i(u_i, x_i, y_i)$ is upper semicontinuous on $D_i \times K_i$;
(v) there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ satisfying $\tilde{y}_i \in A_i(x)$ and $\psi_i(u_i, x_i, \tilde{y}_i) \in -\text{int } C_i(x)$ for all $u_i \in T_i(x)$.

Then (SGIVQVIP) has a solution.

Proof. Although it is similar to the proof of Corollary 1 in [5], we include it for the sake of completeness of the paper.

For each $i \in I$, we set

$$F_i(x, y_i) = \psi_i(T_i(x), x_i, y_i) = \{\psi_i(u_i, x_i, y_i) : u_i \in T_i(x)\}$$

for all $x \in K$ and $y_i \in K_i$. Then, all the conditions of Theorem 3.1 can easily be verified except for condition (iii). Hence we only need to prove that the set

$$\mathcal{D} = \{x \in K : \exists u_i \in T_i(x) \text{ s.t. } \psi_i(u_i, x_i, y_i) \notin -\text{int } C_i(x)\}$$

is closed in K for all $y_i \in K_i$. We prove it for a fixed i.

Let $\{x_\lambda\}$ be a net in $\mathcal{D}$ such that x_λ converges to $x^* \in K$. Then

$$\exists u_{i_\lambda} \in T_i(x_\lambda) \ \text{ s.t. } \ \psi_i(u_{i_\lambda}, x_{i_\lambda}, y_i) \notin -\text{int } C_i(x_\lambda),$$

where x_{i_λ} is the ith component of x_λ, and therefore

$$\psi_i(u_{i_\lambda}, x_{i_\lambda}, y_i) \in W_i(x_\lambda).$$

Let $\mathcal{K} = \{x_\lambda\} \cup \{x^*\}$. Then $\mathcal{K}$ is compact and $u_{i_\lambda} \in T_i(\mathcal{K})$ which is also compact. Therefore u_{i_λ} has a convergent subnet with limit u_{i_*}. Without loss of generality, we may assume that $\{u_{i_\lambda}\}$ converges to u_{i_*}. Then by upper semicontinuity of T, we have $u_{i_*} \in T_i(x^*)$. Since $\psi_i(.,.,y_i)$ is continuous and the graph of W_i is closed, we have

$$\psi_i(u_{i_\lambda}, x_{i_\lambda}, y_i) \text{ converges to } \psi_i(u_{i_*}, x_{i_*}, y_i) \in W_i(x^*),$$

and hence $\psi_i(u_{i_*}, x_{i_*}, y_i) \notin -\text{int } C_i(x^*)$. Therefore, $x^* \in \mathcal{D}$ and thus $\mathcal{D}$ is closed in K. This completes the proof. □

Corollary 3.2. For each $i \in I$, let $K_i, X_i, D_i, \psi_i, T_i$ and W_i be the same as in Corollary 3.1 and let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Assume that the conditions (i) - (iv) of Corollary 3.1 hold. Then (SGIVQVI P) has a solution.

Let W and Z be Hausdorff topological vector spaces and σ be the family of all bounded subsets of W whose union is total in W, that is, the linear hull of $\bigcup\{U : U \in \sigma\}$ is dense in W. Let $\mathcal{B}$ be a neighborhood base of 0 in Z. When U runs through σ, V through $\mathcal{B}$, the family

$$M(U,V) = \{\xi \in L(W,Z) : \cup_{x \in U} \langle \xi, x \rangle \subseteq V\}$$

is a neighborhood base of 0 in $L(W,Z)$ for a unique translation-invariant topology, called the *topology of uniform convergence* on the sets $U \in \sigma$, or, briefly, the σ-topology (see [24], pp. 79-80).

In order to derive existence results for solutions of the (SGVQVLIP) and (SGVQVIP) from Corollary 1, we need the following useful result due to Ding and Tarafdar [16].

Lemma 3.1. Let W and Z be real Hausdorff topological vector spaces and $L(W,Z)$ be the topological vector space under the σ-topology. Then, the bilinear mapping $\langle .,. \rangle : L(W,Z) \times W \to Z$ is continuous on $L(W,Z) \times W$.

Next we derive the existence results for a solution of (SGVQVLIP) by using Corollaries 3.1 and 3.2.

Corollary 3.3. For each $i \in I$, let Y_i be a Hausdorff topological vector space and let K_i, X_i, D_i, T_i and W_i be the same as in Corollary 3.1. For each $i \in I$, let $L(X_i, Y_i)$ be equipped with the σ-topology. For each $i \in I$, let $\eta_i : K_i \times K_i \to X_i$ be affine in the first argument and continuous in the second argument such that $\eta_i(x_i, x_i) = 0$ for all $x_i \in K_i$. Assume that there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ satisfying $\tilde{y}_i \in A_i(x)$ and $\langle u_i, \eta_i(\tilde{y}_i, x_i) \rangle \in -\text{int } C_i(x)$ for all $u_i \in T_i(x)$. Then (SGVQVLIP) has a solution.

Corollary 3.4. For each $i \in I$, let $K_i, X_i, Y_i, D_i, \eta_i, T_i$ and W_i be the same as in Corollary 3.3 and let $L(X_i, Y_i)$ be equipped with the σ-topology. For each $i \in I$, let $\eta_i : K_i \times K_i \to X_i$ be affine in the first argument and continuous in the second argument such that $\eta_i(x_i, x_i) = 0$ for all $x_i \in K_i$. Let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Then (SGVQVIP) has a solution.

In the last of this section, we have the following existence results for a solution of (SGVQVIP).

Corollary 3.5. For each $i \in I$, let K_i, X_i, Y_i, D_i, T_i and W_i be the same as in Corollary 3.3 and let $L(X_i, Y_i)$ be equipped with the σ-topology. Assume that there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ satisfying $\tilde{y}_i \in A_i(x)$ and $\langle u_i, \tilde{y}_i - x_i \rangle \in -\text{int } C_i(x)$ for all $u_i \in T_i(x)$. Then (SGVQVIP) has a solution.

Corollary 3.6. For each $i \in I$, let K_i, X_i, Y_i, D_i, T_i and W_i be the same as in Corollary 3.3 and let $L(X_i, Y_i)$ be equipped with the σ-topology. Let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Then (SGVQVIP) has a solution.

4. Applications

Throughout this section, unless otherwise specified, we assume that the index set I is finite, that is, $I = \{1, \ldots, n\}$. For each $i \in I$, X_i and Y_i are finite dimensional Euclidean spaces $\mathbb{R}^{p_i}$ and $\mathbb{R}^{q_i}$, respectively, and K_i be a nonempty convex subset of X_i. For each $i \in I$, let $C_i : K \to 2^{Y_i}$ be a multivalued map such that for all $x \in K$, $C_i(x)$ is a proper, closed and convex cone with $\text{int } C_i(x) \neq \emptyset$ and $\mathbb{R}^{q_i}_+ \subseteq C_i(x)$. Let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ be defined as $A(x) = \prod_{i \in I} A_i(x)$, for all $x \in K$. For each $i \in I$, let $\varphi_i : K \to_i Y$ be a given vector-valued function. We consider the following *system of vector quasi-optimization problems* (in short, SVQOP) which is to find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and for each $i \in I$,

$$\varphi_i(y) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x}) \quad \text{for all } y \in A(\bar{x}),$$

where $\varphi_i(x) = (\varphi_{i_1}(x), \varphi_{i_2}(x), \ldots, \varphi_{i_{q_i}}(x))$ and for each $l \in \mathcal{L} = \{1, \ldots, q_i\}$, $\varphi_{i_l} : K \to \mathbb{R}$ is a function.

We can choose $y \in A(x)$ in such a way that $y^i = \bar{x}^i$. Then (SVQOP) reduces to the *Debreu type equilibrium problem for vector-valued functions* which is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x}) \quad \text{for all } y_i \in A_i(\bar{x}).$$

It is clear that every solution of (SVQOP) is also a solution of the Debreu type equilibrium problem for vector-valued functions, but the converse need not be true.

Now we recall some definitions.

Definition 4.1. A real-valued function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *locally Lipschitz* if for any $z \in \mathbb{R}^p$ there exist a neighborhood $N(z)$ of z and a positive constant k such that

$$|f(x) - f(y)| \leq k\|x - y\| \quad \text{for all } x, y \in N(z).$$

The Clarke *generalized directional derivative* [9] of a locally Lipschitz function f at x in the direction d denoted by $f^0(x; d)$ is

$$f^0(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}.$$

The Clarke *generalized gradient* [9] of a locally Lipschitz function f at x is defined as

$$\partial f(x) = \{\xi \in \mathbb{R}^p : f^0(x; d) \geq \langle \xi, d \rangle \text{ for all } d \in \mathbb{R}^p\}.$$

If f is convex, then the Clarke generalized gradient coincides with the subdifferential of f in the sense of convex analysis [23].

The generalized invex function was introduced by Craven [10] as a generalization of invex functions [18].

Definition 4.2. A locally Lipschitz function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *generalized invex at x w.r.t. a given function* $\eta : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ if

$$f(y) - f(x) \geq \langle \xi, \eta(y,x) \rangle \quad \text{for all } \xi \in \partial f(x) \text{ and } y \in \mathbb{R}^p.$$

For each $i \in I$, let $\phi_i : K \to \mathbb{R}$ be a locally Lipschitz function and let $x \in K$, $x_j \in K_j$. Following Clarke [9], the *generalized directional derivative at x_j in the direction $d_j \in K_j$* of the function $\phi_i(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)$ denoted by $\phi^0_{ij}(x; d_j)$ is

$$\phi^0_{ij}(x; d_j) = \lim_{\substack{y_j \to x_j \\ t \downarrow 0}} \sup \frac{1}{t} \{ \phi_i(x_1, \ldots, x_{j-1}, y_j + td_j, x_{j+1}, \ldots, x_n)$$

$$- \phi_i(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n) \}.$$

The *partial generalized gradient* [9] of the function $\phi_i(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)$ at x_j is defined as follows:

$$\partial_j \phi_i(x) = \{ \xi_j \in X_j : \phi^0_{ij}(x; d_j) \geq \langle \xi_j, d_j \rangle \text{ for all } d_j \in K_j \}.$$

Lemma 4.1. [9] For each $i \in I$, let $\phi_i : K \to \mathbb{R}$ be locally Lipschitz. Then for each $i \in I$, the multivalued map $\partial_i \phi_i$ is upper semicontinuous.

Definition 4.3. For each $i \in I$, $\phi_i : K \to \mathbb{R}$ is called *generalized invex* at x w.r.t. a given function $\eta_i : K_i \times K_i \to \mathbb{R}^{p_i}$ if

$$\phi_i(y) - \phi_i(x) \geq \langle \xi_i, \eta_i(y_i, x_i) \rangle \quad \text{for all } \xi_i \in \partial_i \phi_i(x) \text{ and } y \in K.$$

Proposition 4.1. For each $i \in I$ and for all $l \in \mathcal{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be generalized invex w.r.t. $\eta_{i_l} : K_i \times K_i \to X_i$. Then any solution of (SGVQVLIP) is a solution of (SVQOP) with $T_i(x) = \partial_i \varphi_i(x)$ for each $i \in I$ and for all $x \in K$ where $\partial_i \varphi_i(x) = (\partial_i \varphi_{i_1}(x), \partial_i \varphi_{i_2}(x), \ldots, \partial_i \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{p_i \times q_i}$.

Proof. Although it is similar to the proof of Proposition 4.1 in [5], we include it for the sake of completeness of the paper.

For the sake of simplicity, we denote by $\varphi_i(x) = (\varphi_{i_1}(x), \ldots, \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{q_i}$, $u_i = (u_{i_1}, \ldots, u_{i_{q_i}})$ where $u_{i_l} \in \partial_i \varphi_{i_l}(x)$ for all $l \in \mathcal{L}$, and

$$\langle u_i, \eta_i(y_i, x_i) \rangle = (\langle u_{i_1}, \eta_{i_1}(y_i, x_i) \rangle, \ldots, \langle u_{i_{q_i}}, \eta_{i_{q_i}}(y_i, x_i) \rangle) \in \mathbb{R}^{q_i}.$$

Assume that $\bar{x} \in K$ is a solution of (SGVVLIP). Then for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}), \ \exists \bar{u}_{i_l} \in \partial_i \varphi_{i_l}(\bar{x}) \text{ for all } l \in \mathcal{L} \text{ such that}$$

$$(\langle \bar{u}_{i_1}, \eta_{i_1}(y_i, \bar{x}_i) \rangle, \ldots, \langle \bar{u}_{i_{q_i}}, \eta_{i_{q_i}}(y_i, \bar{x}_i) \rangle) \notin -\text{int } C_i(\bar{x}).$$

We can rewrite this as

$$\forall y_i \in A_i(\bar{x}), \ \exists \bar{u}_i \in \partial_i \varphi_i(\bar{x}) \ : \ \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i) \rangle \notin -\text{int } C_i(\bar{x}). \qquad (*)$$

Since for each $i \in I$ and for all $l \in \mathcal{L}$, φ_{i_l} is generalized invex w.r.t. η_{i_l}, we have

$$\varphi_{i_l}(y) - \varphi_{i_l}(\bar{x}) \geq \langle u_{i_l}, \eta_{i_l}(y_i, \bar{x}_i) \rangle \quad \text{for all } u_{i_l} \in \partial_i \varphi_{i_l}(\bar{x}) \text{ and } y \in A(\bar{x}),$$

that is, for each $i \in I$

$$\varphi_i(y) - \varphi_i(\bar{x}) \geq \langle u_i, \eta_i(y_i, \bar{x}_i) \rangle \quad \text{for all } u_i \in \partial_i \varphi_i(\bar{x}) \text{ and } y \in A(\bar{x}).$$

Therefore for each $i \in I$ and for all $u_i \in \partial_i \varphi_i(\bar{x})$, we have

$$\begin{aligned} \varphi_i(y) - \varphi_i(\bar{x}) &\in \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i) \rangle + \mathbb{R}^{q_i}_+ \\ &\subseteq \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i) \rangle + \text{int } C_i(\bar{x}). \end{aligned} \qquad (**)$$

From (*) and (**), we have $\varphi_i(y) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x})$. Hence $\bar{x} \in K$ is a solution of (SVQOP). □

Rest of the paper, unless otherwise specified, $\partial_i \varphi_i(x)$ and $\langle u_i, \eta_i(y_i, x_i) \rangle$ are the same as defined in Proposition 4.1.

Theorem 4.1. For each $i \in I$ and for all $l \in \mathcal{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be generalized invex w.r.t. $\eta_{i_l} : K_i \times K_i \to X_i$ such that η_{i_l} is affine in the first argument, continuous in the second argument and $\eta_{i_l}(x_i, x_i) = 0$ for all $x_i \in K_i$. Assume that there exists $r > 0$ such that for all $x \in K$, $\|x\| > r$, there exist $i \in I$ and $\tilde{y}_i \in K_i$ with $\|\tilde{y}_i\|_i \leq r$ satisfying $\tilde{y}_i \in A_i(x)$ and

$$\langle u_i, \eta_i(\tilde{y}_i, x_i) \rangle \in -\text{int } C_i(x) \quad \text{for all } u_i \in \partial_i \varphi_i(x),$$

where $\|\cdot\|$ and $\|\cdot\|_i$ denote the norms on X and X_i, respectively. Then (SVQOP) has a solution.

Proof. For each $i \in I$ and for all $x \in K$, $T_i(x) = \partial_i \varphi_i(x)$ is an upper semicontinuous multivalued map by Lemma 4.1. It is easy to check that all the conditions of Corollary 3.3 are satisfied. Hence from Corollary 3.3 and Proposition 4.1 it follows that (SVQOP) has a solution. □

Theorem 4.2. For each $i \in I$ and for all $l \in \mathcal{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be generalized invex w.r.t. $\eta_{i_l} : K_i \times K_i \to X_i$ such that η_{i_l} is affine in the first argument, continuous in the second argument and $\eta_{i_l}(x_i, x_i) = 0$ for all $x_i \in K_i$. Let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Then (SVQOP) has a solution.

In the next three corollaries, we set $\varphi_i(x) = (\varphi_{i_1}(x), \ldots, \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{q_i}$, $u_i = (u_{i_1}, \ldots, u_{i_{q_i}})$, $\langle u_i, y_i - x_i \rangle = (\langle u_{i_1}, y_i - x_i \rangle, \ldots, \langle u_{i_{q_i}}, y_i - x_i \rangle) \in \mathbb{R}^{q_i}$ and $\partial_i \varphi_i(x) = (\partial_i \varphi_{i_1}(x), \partial_i \varphi_{i_2}(x), \ldots, \partial_i \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{p_i \times q_i}$, where $\partial_i \varphi_{i_j}(x)$ $(j = 1, \ldots, q_i)$ is the partial subdifferential in the sense of convex analysis.

Corollary 4.1. For each $i \in I$ and for all $l \in \mathcal{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be convex and lower semicontinuous Assume that there exists $r > 0$ such that for all $x \in K$, $\|x\| > r$, there exist $i \in I$ and $\tilde{y}_i \in K_i$ with $\|\tilde{y}_i\|_i \leq r$ satisfying $\tilde{y}_i \in A_i(x)$ and

$$\langle u_i, \tilde{y}_i - x_i \rangle \in -\text{int } C_i(x) \quad \text{for all } u_i \in \partial_i \varphi_i(x),$$

where $\|\cdot\|$ and $\|\cdot\|_i$ denote the norms on X and X_i, respectively. Then (SVQOP) has a solution.

Corollary 4.2. For each $i \in I$ and for all $l \in \mathcal{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be convex and lower semicontinuous on K. Let the multivalued map $A = \prod_{i\in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i\in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Then (SVQOP) has a solution.

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