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Existence of a Solution and Variational Principles for Vector Equilibrium Problems[1]

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Abstract. In this paper, we prove an existence result for a solution to the vector equilibrium problems. Then, we establish variational principles, that is, vector optimization formulations of set-valued maps for vector equilibrium problems. A perturbation function is involved in our variational principles. We prove also that the solution sets of our vector optimization problems of set-valued maps contain or coincide with the solution sets of vector equilibrium problems.

Key Words. Existence of a solution, variational principles, vector optimization problems, vector equilibrium problems, set-valued maps.

1. Introduction and Preliminaries

The classical variational inequality (Ref. 1) for vector-valued functions is known as a vector variational inequality (VVI); it has been introduced in 1980 (see Ref. 2) with further applications in finite-dimensional spaces. Since then, the theory of vector variational inequalities has emerged as a new direction for researchers. Because of the applications, it has been studied and generalized by many authors; see for example Ref. 3 and references therein. Recently, Chen et al. (Ref. 4) introduced a gap function for VVI in order to convert a VVI problem into a vector optimization problem (VOP); the significance of their work is that one can study the VVI via the VOP.

Another generalization of the classical variational inequality is the equilibrium problem (Refs. 5–19), which includes as special cases various

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problems, for example complementarity problems, optimization problems, and saddle-point problems. Inspired by early results on this field, many authors considered and studied the vector equilibrium problem (VEP), that is, the equilibrium problem for vector-valued functions; see for example Refs. 3, 14–15, 20–27, and references therein.

We remark that Auchmuty (Ref. 28) proposed variational principles which generalize the concept of gap function (Ref. 29–30) for the variational inequality. Later, Blum and Oettli (Ref. 31) proposed also variational principles for equilibrium problems. Motivated by the recent prosperous development of the VVI and the VEP, our aim in this paper is to contribute to the VEP literature. We derive an existence result for a solution of the VEP. We propose also variational principles for the VEP by using as perturbation function. The work of this paper is a continuation of the work in Ref. 32.

Let X and Y be topological vector spaces, and let C be a pointed closed convex proper cone in Y with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ denotes the interior of the set C. Then C induces a vector ordering in Y setting, for all $x, y \in C$,

$$x \leq y, \qquad \text{if and only if } y - x \in C,$$
$$x \nleq y, \qquad \text{if and only if } y - x \notin C.$$

Since $\operatorname{int} C \neq \emptyset$, we have also a weak order in Y setting, for all $x, y \in C$,

$$x < y, \qquad \text{if and only if } y - x \in \operatorname{int} C,$$
$$x \nless y, \qquad \text{if and only if } y - x \notin \operatorname{int} C.$$

The orderings $\geq$, $\ngeq$, $>$, $\ngtr$ are defined similarly.

Let K be a nonempty convex set in X, and let f: $K \times K \to Y$ be a bifunction such that

$$f(x, x) = 0, \qquad \text{for all } x \in K.$$

Then, we consider the vector equilibrium problem (VEP), which is to find $x_0 \in K$ such that

$$f(x_0, y) \nless 0, \qquad \text{for all } y \in K. \tag{1}$$

The solution set of this problem will be denoted by P.

Existence of a solution of this problem is investigated in Refs. 14–15 and 21–22. For the case of a moving cone, it is studied also in Refs. 20 and 24. For a direct application of the VEP, we refer to Ref. 23.

A function q: $K \to Y$ is called quasiconvex (Ref. 22) if, for all $\alpha \in Y$, the set

$$U_{\leq}(\alpha) = \{x \in K: q(x) \leq \alpha\}$$

is convex. If q is quasiconvex, then the set

$$U_{<}(\alpha) = \{x \in K: q(x) < \alpha\}$$

is convex also (see Ref. 22).

A function $q: K \to Y$ is called upper semicontinuous (Refs. 33–34) on K if, for all $\alpha \in Y$, the (upper level) set

$$U(\alpha) = \{x \in K: q(x) \nless \alpha\}$$

is closed in K.

If A is a nonempty subset of Y, then we set

$$\max_C A := \{a \in A \mid \text{there exists no } a' \in A \text{ such that } a' \neq a \text{ and } a' \geq a\},$$

$$\min_C A := \{a \in A \mid \text{there exists no } a' \in A \text{ such that } a' \neq a \text{ and } a' \leq a\},$$

$$\mathrm{w} - \max_C A := \{a \in A \mid \text{there exists no } a' \in A \text{ such that } a' > a\},$$

$$\mathrm{w} - \min_C A := \{a \in A \mid \text{there exists no } a' \in A \text{ such that } a' < a\}.$$

Note that it is possible that $\max_C A = \varnothing$.

A point-to-set map $T: X \rightrightarrows X$ is called a KKM-map if, for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X,

$$\mathrm{conv}(\{x_1, x_2, \ldots, x_n\}) \subseteq \bigcup_{i=1}^{n} T(x_i),$$

where $\mathrm{conv}(A)$ denotes the convex hull of a nonempty set $A \subseteq X$.

We shall use the following well-known Fan-KKM theorem to prove the existence of a solution to the VEP.

Lemma 1.1. See Ref. 35. Let A be an arbitrary set in a topological vector space X. Let $T: A \rightrightarrows X$ be a KKM-map such that $T(x)$ is closed for all $x \in A$ and is compact for at least one $x \in A$. Then $\bigcap_{x \in A} T(x) \neq \varnothing$.

2. Existence Result

For all $A \subset X$, we denote by $\mathrm{cl}_X(A)$ the closure of A in X and by $\mathscr{F}(X)$ the family of all nonempty finite subsets of X.

In this section, we prove the following existence theorem for a solution to the VEP.

Theorem 2.1. Let K be a nonempty convex subset of a topological vector space X and, for each $y \in K$, let $x \mapsto f(x, y)$ be upper semicontinuous on each nonempty compact subsets of K. Assume that there exists a function

$p: K \times K \to Y$ such that:

(a) for each $x, y \in K, f(x, y) < 0$ implies $p(x, y) < 0$,
(b) for each $A \in \mathscr{F}(X)$ and each $x \in \mathrm{co}(A)$, $y \mapsto p(x, y)$ is quasiconvex,
(c) for each $x \in K, p(x, x) \nless 0$,
(d) there exist a nonempty closed compact subset $D \subseteq K$ and $y^* \in D$ such that $p(x, y^*) < 0$, for all $x \in K \backslash D$.

Then, there exists $x_0 \in D \subseteq K$ such that

$$f(x_0, y) \nless 0, \qquad \text{for all } y \in K.$$

Proof. For each $y \in K$, we define

$$G(y) = \{x \in D: f(x, y) \nless 0\}.$$

For each $y \in K$, since $x \mapsto f(x, y)$ is upper semicontinuous on each nonempty compact subsets of K, we have that each $G(y)$ is closed. Since every element $x_0 \in \bigcap_{y \in K} G(y)$ is a solution of (3), we have to prove that

$$\bigcap_{y \in K} G(y) \neq \varnothing.$$

Since D is compact, it is sufficient to show that the family $\{G(y)\}_{y \in K}$ has the finite intersection property.

Let $\{y_1, y_2, \ldots, y_m\}$ be a finite subset of K. We note that

$$A := \mathrm{conv}(\{y_1, y_2, \ldots, y_m\})$$

is a compact convex subset of K. We define a point-to-set map $F: A \rightrightarrows A$ as

$$F(y) = \{x \in A: p(x, y) \nless 0\}, \qquad \text{for all } y \in A.$$

By condition (c), $F(y)$ is nonempty. We see that F is a KKM-map.

Suppose that there exists a finite subset $\{v_1, v_2, \ldots, v_n\} \subseteq A$ and scalars $\alpha_i \geq 0$, $i = 1, \ldots, n$, with $\sum_{i=1}^{n} \alpha_i = 1$, such that

$$\sum_{i=1}^{n} \alpha_i v_i \notin \bigcup_{i=1}^{n} F(v_i).$$

Then, we have

$$p\left(\sum_{i=1}^{n} \alpha_i v_i, v_i\right) < 0, \qquad \text{for all } i.$$

By condition (b), we have

$$p\left(\sum_{i=1}^{n} \alpha_i v_i, \sum_{i=1}^{n} \alpha_i v_i\right) < 0,$$

a contradiction to condition (c). Hence, F is a KKM-map. From Condition (a), we have that

$$F(y) \subseteq G(y), \qquad \text{for all } y \in A.$$

We note that, for each $y \in A$, $\mathrm{cl}_A(F(y))$is closed in A and therefore is compact also. By Lemma 1.1,

$$\bigcap_{y \in A} \mathrm{cl}_A(F(y)) \neq \emptyset.$$

We can choose

$$\bar{x} \in \bigcap_{y \in A} \mathrm{cl}_A(F(y)),$$

and note that $y^* \in A$ and $F(y^*) \subseteq D$ by (d). Thus,

$$\bar{x} \in \mathrm{cl}_A(F(y^*)) \subseteq \mathrm{cl}_K(F(y^*)) = \mathrm{cl}_D(F(y^*)) \subseteq D.$$

Since

$$\bar{x} \in \bigcap_{j=1}^{m} \mathrm{cl}_A(F(y_j))$$

and since, for each $j = 1, 2, \ldots, m$,

$$\begin{aligned} \mathrm{cl}_A(F(y_j)) &= \mathrm{cl}_A(\{x \in A : p(x, y_j) \nless 0\}) \\ &\subseteq \mathrm{cl}_A(\{x \in A : f(x, y_j) \nless 0\}) \\ &= \{x \in A : f(x, y_j) \nless 0\}, \end{aligned}$$

we have

$$f(\bar{x}, y_j) \nless 0, \qquad \text{for all } j = 1, 2, \ldots, m,$$

and hence,

$$\bar{x} \in \bigcap_{j=1}^{m} G(y_j).$$

Therefore, $\{G(y)\}_{y \in K}$ has the finite intersection property and the proof is finished. □

Remark 2.1.

(a) Theorem 2.1 can be seen as an extension of Theorem 3.1 in Ref. 36 for vector-valued functions.
(b) In Theorem 2.1, X need not be Hausdorff.

3. Variational Principles

Extending the terminology of Auchmuty (Ref. 28) and of Blum and Oettli (Ref. 31), we say that a variational principle holds for the VEP (1) if there exists a set-valued map $F: K \rightrightarrows Y$, depending on the data of the VEP but not on its solution set, such that the solution set of the VEP coincides with the solution set of the following vector optimization problem (VOP):

$$\mathrm{w} - \min_{x \in K} F(x); \tag{2}$$

that is, to find all $x_0 \in K$ for which there exists $y_0 \in F(x_0)$ such that

$$y_0 \in \mathrm{w} - \min_C F(K);$$

i.e.,

$$F(x_0) \cap \mathrm{w} - \min_C F(K) \neq \emptyset, \qquad \text{where } F(K) = \bigcup_{x \in K} F(x).$$

For the existence of a solution and other theoretical work on the VOP (2), we refer to Refs. 37–41 and references therein.

Let $h: K \times K \to Y$ be a bifunction such that:

(i) $h(x, x) = 0$, for all $x \in K$,
(ii) $h(x, y) > 0$, for all $x \neq y$, $x, y \in K$.

Next, let us consider the following VEP associated to the function $f + h$: find $x_0 \in K$ such that

$$f(x_0, y) + h(x_0, y) \not< 0, \qquad \text{for all } y \in K. \tag{3}$$

We denote by S the solution set of the VEP (3). The existence of a solution to the VEP (3) has been investigated by Ansari (Ref. 20), by Bianchi et al. (Ref. 22), and by Tan and Tinh (Ref. 27). To obtain the equivalence of P and S, we need the following additional condition on h:

(iii) for any $x \in K$, $y \in K$, and $\alpha \in [0, 1]$, the function $\tau(\alpha) = h(x, \alpha y + (1 - \alpha)x)$ is homogenous with a degree $\sigma > 1$; i.e., $\tau(\alpha) = \alpha^\sigma \tau(1)$.

Proposition 3.1. Suppose that $f(x, \cdot)$ is convex for each $x \in K$ and that h satisfies (i)–(iii). Then, $P = S$.

Proof. Suppose that x_0 solves the VEP (1). If $x_0 \notin S$, then there is $y \in K$ such that

$$f(x_0, y) + h(x_0, y) < 0,$$

and hence,

$$f(x_0, y) < -h(x_0, y) < 0,$$

a contradiction.

Suppose that x_0 solves the VEP (3). For contradiction, assume that $x_0 \notin P$. Then, there is $y \in K$ such that $f(x_0, y) < 0$. Set

$$y_\alpha = \alpha y + (1 - \alpha)x_0.$$

For each $\alpha \in (0, 1)$, we have

$$\begin{aligned} &f(x_0, y_\alpha) + h(x_0, y_\alpha) \\ &\leq \alpha f(x_0, y) + (1 - \alpha) f(x_0, x_0) + \alpha^\sigma h(x_0, y) \\ &= \alpha f(x_0, y) + \alpha^\sigma h(x_0, y). \end{aligned}$$

Note that there exists $\alpha' \in (0, 1)$, such that

$$f(x_0, y) + \alpha^{\sigma - 1} h(x_0, y) < 0,$$

when $\alpha \in (0, \alpha')$. It follows that

$$f(x_0, y_\alpha) + h(x_0, y_\alpha) \leq \alpha(f(x_0, y) + \alpha^{\sigma - 1} h(x_0, y)) < 0,$$

a contradiction, since $y_\alpha \in K$. □

Combining Theorem 2.1 and Proposition 3.1, we have the following existence result for a solution to the VEP (3).

Theorem 3.1. Let the assumptions of Theorem 2.1 and Proposition 3.1 hold. Then, there exists a solution to the VEP (3).

Remark 3.1. In Banach spaces, it suffices to suppose that $\|\tau(\alpha)\| \leq \alpha^\sigma \|\tau(1)\|$.

In order to formulate our variational principle, we define a point-to-set map $G: K \rightrightarrows Y$ as follows:

$$G(x) := \mathrm{w} - \min\{f(x, y) + h(x, y) \mid y \in K\},$$

where h is the same as defined above and can be termed a perturbation bifunction; we associate to the VEP (1) the following VOP for a point-to-set map:

$$\mathrm{w} - \max_{x \in K} G(x); \tag{4}$$

that is, to find all $x_0 \in K$ for which there exists $y_0 \in G(x_0)$ such that

$$y_0 \in \mathrm{w} - \max_C G(x);$$

i.e.,

$$G(x_0) \cap \mathrm{w} - \max_C G(K) \neq \emptyset, \qquad \text{where } G(K) = \bigcup_{x \in K} G(x).$$

We denote by Q the solution set of the VOP (4).

Using Theorem 3.1 from Ref. 32 with $f = f + h$, we obtain the following result.

Proposition 3.2.

(i) $x_0 \in K$ is a solution of the VEP (3) if and only if $0 \in G(x_0)$.
(ii) $S \subseteq Q$.

Now, by combining Propositions 3.1 and 3.2, we obtain the following result.

Theorem 3.2. Let the assumptions of Proposition 3.1 hold. Then:

(i) $\bar{x} \in K$ is a solution of the VEP (1) if and only if $0 \in G(\bar{x})$.
(ii) $P \subseteq Q$.

Now, we consider another VOP whose solution set coincides with the solution set of the VEP. Let Y be a locally convex space; then, we can define the dual cone C^* to C. Since $\mathrm{int}\, C \neq \emptyset$ and $C \neq Y$, we have $C^* \neq \{0\}$; also, C^* has a weakly* compact base; i.e., there exists $B \subseteq C^*$, B is convex, weakly* compact, such that $0 \notin B$ and $C^* = \bigcup_{t \geq 0} tB$. We fix such a base and set

$$\sigma(u) := \max_{t \in B}(t, u), \qquad \text{for all } u \in Y;$$

see Ref 25. Then, for all $u \in Y$,

$$u < 0 \Leftrightarrow \sigma(u) < 0,$$

$$u \leq 0 \Leftrightarrow \sigma(u) \leq 0,$$

$$u \nless 0 \Leftrightarrow \sigma(u) \geq 0,$$

$$u \nleq 0 \Leftrightarrow \sigma(u) > 0.$$

In order to formulate our second variational principle, we define the set-valued maps $Z: K \rightrightarrows K$ and $\Phi: K \rightrightarrows Y$ as follows:

$$Z(x) := \{y \in K \,|\, \sigma(f(x, y) + h(x, y)) \leq \sigma(f(x, z) + h(x, z)), \forall z \in K\},$$

$$\Phi(x) := f(x, Z(x)) + h(x, Z(x)).$$

Now, we define the following VOP for a set-valued map:

$$\max_{x \in K} \Phi(x); \tag{5}$$

that is, to find all $x_0 \in K$ for which there exists $y_0 \in \Phi(x_0)$ such that $y_0 \in \max_C \Phi(K)$: i.e.,

$$\Phi(x_0) \cap \max_C \Phi(K) \neq \emptyset.$$

We denote by R the solution set of problem (5).

Applying Theorem 3.2 from Ref. 32 with $f = f + h$, we obtain the following result.

Proposition 3.3.

(i) $x_0 \in K$ is a solution of the VEP (3) if and only if $0 \in \Phi(x_0)$.
(ii) If the solution of the VEP (3) is nonempty, then $S = R$.

By combining Propositions 3.1 and 3.3, we have the following result.

Theorem 3.3. Let the assumptions of Proposition 3.1 hold. Then:
(i) $\bar{x} \in K$ is a solution of the VEP (1) if and only if $0 \in \Phi(\bar{x})$.
(ii) If the solution set of the VEP (1) is nonempty, then $P = R$.

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