Indian J. pure appl. Math., 30(2) : 161-166, February 1999

VARIATIONAL-LIKE INEQUALITIES FOR MULTIVALUED MAPS

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(*Received* 28 *January* 1998; *accepted* 14 *July* 1998)

In this paper, we introduce a more general form of variational-like inequalities for multivalued maps and prove the existence of its solution in the setting of reflexive Banach spaces.

Key Words : Inequalities — Variational like; Multivalued Maps; Banach spaces; Nonempty subsets; Convex mathematical programming

1. INTRODUCTION

Let X be a reflexive Banach space with its dual X^* and, K and C be nonempty subsets of X and X^*, respectively. Given two maps $M : K \times C \to X^*$ and $\eta : K \times K \to X$, and a multivalued map $T : K \to 2^C$, then we consider the following problem:

Problem 1 — Find $x_0 \in K$ such that for each $y \in K$, $\exists\, u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \eta(y, x_0) \rangle + b(x_0, y) - b(x_0, x_0) \geq 0, \quad \dots (1.1)$$

where $b : K \times K \to I\!R$ is not necessarily differentiable and satisfies some proper conditions, and $\langle \cdot, \cdot \rangle$ is the pairing between X^* and X.

If $b \equiv 0$, then Problem 1 reduces to the problem of finding $x_0 \in K$ such that for each $y \in K$, $\exists\, u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \eta(y, x_0) \rangle \geq 0. \quad \dots (1.2)$$

This poblem is the weak formulation of generalized variational-like inequality problem (GVLIP), introduced by Parida and Sen[3] in finite dimensional spaces. They have also shown the relationship betwen (GVLIP) and convex mathematical programming. It has been further studied by Yao[5,6] with applications in complementarity problems.

If we take $M(x, u) = u$ and $\eta(x, y) = g(y) - g(x)$, $\forall x, y \in K$, where $g : K \to K$ then Problem 1 is equivalent to find $x_0 \in K$ such that for each $y \in K$, $\exists\, u_0 \in T(x_0)$ such that

$$\langle u_0, g(y) - g(x_0) \rangle + b(x_0, y) - b(x_0, x_0) \geq 0. \quad \dots (1.3)$$

Such problem was introduced and studied by Ding and Tarafdar[1] in the setting of locally convex Hausdorff topological vector spaces.

If $M(x, u) = u$ and $b(x, y) = h(y)$, $\forall x \in K$ then Problem 1 becomes to the problem of finding $x_0 \in K$ such that for each $y \in K$, $\exists u_0 \in T(x_0)$ such that

$$\langle u_0, \eta(y, x_0) \rangle + h(y) - h(x_0) \geq 0. \quad \ldots (1.4)$$

It has been introduced and studied by Siddiqi, Ansari and Ahmad[4].

In this paper, we prove the existence of solution of Problem 1, which is more general and unifying one. We also derive the existence theorem for a special case of Problem 1.

We need the following concept and result for the proof of our main result. We denote *conv* (A), $\forall A \subset X$, the convex hull of A.

Definition 1.1 — A map $T : X \to 2^X$ is called KKM-map, if for every finite subset $\{x_1, \ldots, x_n\}$ of X, $conv\,(\{x_1, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} T(x_i)$.

Lemma 1.1 (KKM-FAN[2]) — Let A be an arbitrary nonempty set in a topological vector space E and $T : A \to 2^X$ be a *KKM-map*. If $T(x)$ is closed for all $x \in A$ and is compact for at least one $x \in A$ then $\bigcap_{x \in A} T(x) \neq 0$.

2. Existence Results

First, we give some definitions which are necessary for the proof of existence theorem for Problem 1.

Definition 2.1 — Let X be a normed space with its dual X^*, C be a nonempty subset of X^* and K be a nonempty convex subset of X. Given two maps $M : K \times C \to X^*$ and $\eta : K \times K \to X$, then a multivalued map $T : K \to 2^C$ is called :

(*i*) η-monotone with respect to M if for every pair of points $x \in K, y \in K$ and for all $u \in T(x), v \in T(y)$ such that $\langle M(x, u) - M(y, v), \eta(x, y) \rangle \geq 0$; and

(*ii*) *V*-hemicontinuous with respect to M if $\forall x, y \in K$, $\alpha \geq 0$ and $u_\alpha \in T(x + \alpha y)$, there exists $u_0 \in T(x)$ such that for any $z \in K$, $\langle M(x, u_\alpha), z \rangle \to \langle M(x, u_0), z \rangle$ as $\alpha \to 0^+$.

Remark 2.1 : If $M(x, u) = u$ and $\eta(y, x) = y - x$, $\forall x, y \in K$ then above definitions (*i*) and (*ii*) reduce to the definitions of monotonicity and *V*-hemicontinuity of T, respectively.

Now we prove the main result of this paper.

Theorem 2.1 — *Assume that*

1° *K is a nonempty closed bounded convex subset of a reflexive Banach space X;*

2° *C is a nonempty subset of X^*;*

3° *$M : K \times C \to X^*$ is continuous and affine in the first argument;*

4° *$\eta : K \times K \to X$ is continuous and affine in both the argument such that $\eta(x, x) = 0$, $\forall x \in K$*

5° *$T : K \to 2^C$ is η-monotone and V-hemicontinuous with respect to M such that $T(x)$ is compact, $\forall x \in K$;*

If $M(x, u) = u$ and $b(x, y) = h(y)$, $\forall x \in K$ then Problem 1 becomes to the problem of finding $x_0 \in K$ such that for each $y \in K$, $\exists u_0 \in T(x_0)$ such that

$$\langle u_0, \eta(y, x_0) \rangle + h(y) - h(x_0) \geq 0. \quad \ldots (1.4)$$

It has been introduced and studied by Siddiqi, Ansari and Ahmad[4].

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3° *$M : K \times C \to X^*$ is continuous and affine in the first argument;*

4° *$\eta : K \times K \to X$ is continuous and affine in both the argument such that $\eta(x, x) = 0$, $\forall x \in K$*

5° *$T : K \to 2^C$ is η-monotone and V-hemicontinuous with respect to M such that $T(x)$ is compact, $\forall x \in K$;*

6° *$b : K \times K \to \mathbb{R}$ is continuous and convex in the second argument;*

7° *the set $\{x \in K : \exists v \in T(y)$ such that $\langle M(y, v), \eta(y, x)\rangle + b(x, y) - b(x, x) \geq 0, \ \forall y \in K\}$, is convex.*

Then there exists a solution of Problem 1.

PROOF : For each $y \in K$, we define

$$F_1(y) = \{x \in K : \exists u \in T(x) \text{ such that } \langle M(x, u), \eta(y, x)\rangle + b(x, y) - b(x, x) \geq 0\}.$$

Then F_1 is a *KKM-map*. Indeed, let $\{x_1, \ldots, x_n\} \subset K, \alpha_i \geq 0 \ \forall i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$ and $\bar{x} = \sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{i=1}^{n} F_1(x_i)$. Then for any $\bar{u} \in T(\bar{x})$, we have

$$\langle M(\bar{x}, \bar{u}), \eta(x_i, \bar{x})\rangle + b(\bar{x}, x_i) - b(\bar{x}, \bar{x}) < 0, \ \forall i = 1, 2, \ldots, n.$$

or

$$\sum_{i=1}^{n} \alpha_i \langle M(\bar{x}, \bar{u}), \eta(x_i, \bar{x})\rangle, + \sum_{i=1}^{n} \alpha_i b(\bar{x}, x_i) - b(\bar{x}, \bar{x}) < 0.$$

Since $\eta(\cdot, \cdot)$ is affine and $b(\cdot, \cdot)$ is convex in the second argument, we have

$$\left\langle M(\bar{x}, \bar{u}), \eta\left(\sum_{i=1}^{n} \alpha_i x_i, \bar{x}\right)\right\rangle + b\left(\bar{x}, \sum_{i=1}^{n} \alpha_i x_i\right) - b(\bar{x}, \bar{x})$$

$$\leq \sum_{i=1}^{n} \alpha_i \langle M(\bar{x}, \bar{u}), \eta(x_i, \bar{x}) + \sum_{i=1}^{n} \alpha_i b(\bar{x}, x_i)(\bar{x}, \bar{x}) < 0.$$

This implies that $\langle M(\bar{x}, \bar{u}), \eta(\bar{x}, \bar{x}) + b(\bar{x}, \bar{x}) - b(\bar{x}, \bar{x}) < 0$. But, since $\eta(x, x) = 0, \ \forall x \in K$, we have

$$\langle M(\bar{x}, \bar{u}), \eta(\bar{x}, \bar{x})\rangle = 0.$$

Therefore, we reach to a contradiction. Hence, F_1 is a *KKM-map*.

Define a multivalued map $F_2 : K \to 2^K$ as, for each $y \in K$,

$$F_2(y) = \{x \in K : \exists v \in T(y) \text{ such that } \langle M(y, v), \eta(y, x)\rangle + b(x, y) - b(x, x) \geq 0\}.$$

Then $F_1(y) \subset F_2(y), \ \forall y \in K$:

Let $x \in F_1(y)$ then $\exists u \in T(x)$ such that

$$\langle M(x, u), \eta(y, x)\rangle + b(x, y) - b(x, x) \geq 0.$$

For all $v \in T(y)$, we have

$$\langle M(y, v) - M(x, u), \eta(y, x)\rangle \leq \langle M(y, v), \eta(y, x)\rangle + b(x, y) - b(x, x).$$

Since T is η-monotone with respect to M, we have

$$\langle M(y, v), \eta(y, x) \rangle + b(x, y) - b(x, x) \geq 0.$$

So, $x \in F_2(y)$. Therefore, $F_1(y) \subset F_2(y)$, $\forall y \in K$ and hence $F_2(y)$ is also a *KKM-map*.

$F_2(y)$, $\forall y \in K$ is closed: Let $\{x_n\}$ be sequence in $F_2(y)$ such that $x_n \to x_0$. Then $x_0 \in K$. Since $x_n \in F_2(y)$ $\forall n$, $\exists v_n \in T(y)$ such that

$$\langle M(y, v_n), \eta(y, x_n) \rangle + b(x_n, y) - b(x_n, x_n) \geq 0.$$

Since $T(y)$ is compact, without loss of generality, we assume that there exists $v_0 \in T(y)$ such that $v_n \to v_0$. Since $M(\cdot, \cdot)$, $\eta(\cdot, \cdot)$, $b(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ are continuous, we have

$$\langle M(y, v_n), \eta(y, x_n) \rangle + b(x_n, y) - b(x_n, x_n) \to \langle M(y, v_0), \eta(y, x_0) + b(x_0, y) - b(x_0, x_0).$$

Therefore, $\langle M(y, v_0), \eta(y, x_0) \rangle + b(x_0, y) - b(x_0, x_0) \geq 0$. So, $x_0 \in F_2(y)$ and hence $F_2(y)$ is closed.

By assumption 7^o, $F_2(y)$ is convex. Now we equip X with weak topology. Then K, as a closed convex subset in the reflexive Banach space X, is weakly compact. Since $F_2(y)$ is a closed convex subset of a reflexive Banach space then $F_2(y)$ is weakly closed. $F_2(y) \subset K$ and weak closedness of $F_2(y)$, we have $F_2(y)$ is weakly compact. Then by Lemma 1.1, we have $\bigcap_{y \in K} F_2(y) \neq 0$.

Let $x \in \bigcap_{y \in K} F_2(y)$. Then for any $y \in K, \exists v_y \in T(y)$ such that

$$\langle M(y, v_y), \eta(y, x) \rangle + b(x, y) - b(x, x) \geq 0.$$

By convexity of K, for any $\alpha \in (0, 1)$ there exists $v_\alpha \in T(\alpha y + (1 - \alpha)x)$ such that

$$\langle M(\alpha y + (1 - \alpha)x, v_y), \eta(\alpha y + (1 - \alpha)x, x) \rangle + b(x, \alpha y + (1 - \alpha)x) - b(x, x) \geq 0.$$

Since $M(\cdot, \cdot)$ and $\eta(\cdot, \cdot)$ are affine in the first argument and $b(\cdot, \cdot)$ is convex in the second argument, we have

$$\alpha^2 \langle M(y, v_\alpha), \eta(y, x) \rangle + \alpha(1 - \alpha) \langle M(y, _\alpha), \eta(x, x) \rangle + \alpha(1 - \alpha) \langle M(x, v_\alpha), \eta)y, x) \rangle$$

$$+ (1 - \alpha)^2 \langle M(x, v_\alpha), \eta(x, x) \rangle + \alpha b(x, y) + (1 - \alpha)\, b(x, x) - b(x, x) \geq 0.$$

Since $\eta(x, x) = 0$, $\forall x \in K$, we have

$$\alpha^2 \langle M(y, v_\alpha), \eta(y, x) \rangle + \alpha(1 - \alpha) \langle M(x, v_\alpha), \eta(y, x) \rangle + \alpha b(x, y) - \alpha b(x, x) \geq 0.$$

Dividing by α, we get

$$\alpha \langle M(y, v_\alpha), \eta(y, x) \rangle + (1 - \alpha) \langle M(x, v_\alpha), \eta(y, x) \rangle + b(x, y) - b(x, x) \geq 0.$$

Taking $\alpha \to 0^+$ and by V-hemicontinuity of T with respect to M, there exists $v_0 \in T(x)$ such that

$$\langle M(x_0, u_0), \eta(y, x_0) \rangle + b(x_0, y) - b(x_0, x_0) \geq 0.$$

Theorem 2.2 — *Assume that*

1° *K is a nonempty closed bounded convex subset of a reflexive Banach space X;*

2° *C is a nonempty subset of X^*;*

3° *$M : K \times C \to X^*$ is continuous and affine in the first argument;*

4° *$\eta : K \times K \to X$ is continuous and affine in both the argument such that $\eta(x, x) = 0$, $\forall\, x \in K$;*

5° *$T : K \to 2^C$ is η-monotone and V-hemicontinuous with respect to M such that $T(x)$ is compact, $\forall x \in K$;*

6° *$h : K \to I\!R$ is convex and lower semicontinuous proper functional.*

Then there exist $x_0 \in K$ such that for each $y \in K$, $\exists\, u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \eta(y, x_0) \rangle + h(y) - h(x_0) \geq 0.$$

PROOF : Take $b(x, y) = h(y)$, $\forall x \in K$ in Theorem 2.1, then the proof follows by the proof of Theorem 2.1, if we prove that the set

$$A = \{x \in K : \exists\, v \in T(y) \text{ such that } \langle M(y, v), \eta(y, x) \rangle + h(y) - h(x) \geq 0, \ \forall\, y \in K\}$$

is convex.

Indeed, let $x_1, x_2 \in A$, $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Then for all $y \in K$, $\exists\, v \in T(y)$ such that

$$\langle M(y, v), \eta(y, x_1) \rangle + h(y) - h(x_1) \geq 0 \qquad \text{... (2.1)}$$

and

$$\langle M(y, v), \eta(y, x_2) \rangle + h(y) - h(x_2) \geq 0. \qquad \text{... (2.2)}$$

Multiplying (2.1) and (2.2) by α and β, respectively and then adding, we get

$$\alpha \langle M(y, v), \eta(y, x_1) \rangle + \beta \langle M(y, v), \eta(y, x_2) \rangle + \alpha h(y) + \beta h(y) - \alpha h(x_1) - h(x_2) \geq 0.$$

Since $\eta(\cdot, \cdot)$ is affine and h is convex, we have

$$\langle M(y, v), \eta(y, \alpha x_1 + \beta x_2) \rangle + h(y) - h(\alpha x_1 - \beta x_2) \geq 0.$$

This implies that $\alpha x_1 + \beta x_2 \in A$ and hence A is convex.

Corollary 2.1 — Assume that

1° K is a nonempty closed bounded convex subset of a reflexive Banach space X;

2° C is a nonempty subset of X^*;

3° $M : K \times C \to X^*$ is continuous and affine in the first argument;

4° $\eta : K \times K \to X$ is continuous and affine in both the argument such that $\eta(x, x) = 0$, $\forall\, x \in K$;

5° $T : K \to 2^C$ is η-monotone and V-hemicontinuous with respect to M such that $T(x)$ is compact, $\forall\, x \in K$.

Then there exist $x_0 \in K$ such that for each $y \in K$, $\exists\, u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \eta(y, x_2) \rangle + \geq 0.$$

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$$| p'(z_0) + mn\, \beta\, z_0^{n-1} | \leq \frac{n}{1 + k^{\mu}} [M(p, 1) + m \,|\, \beta \,|], \qquad \ldots (3.2)$$

where

$$| p'(z_0) | = M(p', 1).$$

Now choosing the argument of β suitably in (3.2) and finally letting $|\beta| \to \frac{1}{k^n}$, Theorem 2 follows.

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