GENERALIZED VECTOR VARIATIONAL–LIKE INEQUALITIES AND THEIR SCALARIZATIONS

Qamrul Hasan Ansari

Department of Mathematics
Aligarh Muslim University, Aligarh, India
and Department of Applied Mathematics
National Sun Yat–sen University
Kaohsiung, Taiwan, R.O. China

Abul Hasan Siddiqi

Department of Mathematics
Aligarh Muslim University, Aligarh, India
and Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran, Saudi Arabia

Jen–Chih Yao

Department of Applied Mathematics,
National Sun Yat–sen University
Kaohsiung, Taiwan, R.O. China

ABSTRACT In this paper, we consider a more general form of vector variational-like inequalities for multivalued maps and prove some results on the existence of solutions of our new class of vector variational-like inequalities in the setting of topological vector spaces. Several special cases were also discussed.

KEY WORDS Generalized Vector Variational–Like Inequality, V–hemicontinuous maps, $C_z - \eta$–pseudomonotone maps, Scalarization.

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1. INTRODUCTION

The Vector Variational Inequality (for short, VVI) has been introduced in [16] in the setting of finite dimensional Euclidean space. Since then, the VVI has been studied by Chen et al [6-8, 10], Fu [15], Lai and Yao [19], Lee et al [22], Siddiqi et al. [29], Yang [33-36] and, Yu and Yao [39] in abstract spaces. Later on, the VVI has been extended and generalized in many different directions. Motivations for this come from the fact that the VVI and its generalizations have applications in Optimization, Mathematical Programming, Operations Research and Economics. The Vector Variational-Like Inequality (for short, VVLI), a generalized form of the VVI, was studied by Lin [26], Siddiqi et al. [30] and Yang [32] with applications in Vector Optimization. VVI and VVLI for multivalued maps, considered by Chen and Craven [9], Daniilidis and Hadjisavvas [13], Konnov and Yao [18], Lee et al [20-21, 23-24], Lin et al [27], Ansari [1-3], Ansari and Siddiqi [4] and Lee et al [25], have been shown to be a powerful tool to solve problems from Vector Optimization. Inspired and motivated by the applications of the VVLI for multivalued maps, in this paper we consider a more general form of the VVLI for multivalued maps and introduce the concept of $C_z-\eta$-pseudomonotone multivalued maps. By using a fixed point theorem of Chowdhury and Tan [11], which is a generalized form of Fan-Browder fixed point theorem [5,14], we prove some results on the existence of solutions of our new class of the VVLI for multivalued maps in the setting of topological vector spaces. Several special cases are discussed.

Let X and Y be topological vector spaces, and X^* be the topological dual of X. Let K and D be nonempty subsets of X and X^*, respectively. Let $L(X,Y)$ be the space of all continuous linear maps from X into Y, and let $\langle L(X,Y), X\rangle$ be a dual system of $L(X,Y)$ and X. Let $C : K \rightrightarrows Y$ be a multivalued map, such that $\forall x \in K$, $C(x)$ is a proper, closed and convex cone in Y with apex at the origin and with int $C(x) \neq \emptyset$, where int $C(x)$ denotes the interior of $C(x)$. Given two maps $\theta : K \times D \to L(X,Y)$ and $\eta : K \times K \to X$, and a multivalued map $T : K \rightrightarrows D$, we consider the following Generalized Vector Variational-Like Inequality (for short, GVVLI) which consists in finding $y \in K$, such that

$$\forall x \in K,\ \exists v \in T(y) \text{ such that } \langle \theta(y,v), \eta(x,y)\rangle \not\geq_{\operatorname{int} C(y)} 0,$$

where the inequality means that $\langle \theta(y,v), \eta(x,y)\rangle \notin \operatorname{int} C(y)$.

A point $y \in K$ is said to be a *strong solution* of the GVVLI, iff $\exists v \in T(y)$, such that

$$\langle \theta(y,v), \eta(x,y) \rangle \not\geq_{C(y)} 0 \quad , \quad \forall x \in K .$$

Obviously, every strong solution is a solution of the GVVLI, but in general the reverse claim is false.

Let $\tilde{C} : K \rightrightarrows Y$ be a multivalued map such that $\forall x \in K$, $\tilde{C}(x)$ is a proper, closed and convex cone in Y with apex at the origin and with int $\tilde{C}(x) \neq \emptyset$.

When $\theta(y,v) = Av$, where $A : D \to L(X,Y)$ is a nonlinear map, and $C(y) = -\tilde{C}(y)$, the GVVLI reduces to the following Extended Generalized Vector Variational–Like Inequality which consists in finding $y \in K$, such that

$$\forall x \in K, \ \exists v \in T(y) \ : \ \langle Av, \eta(x,y) \rangle \not\geq_{-\mathrm{int}\, \tilde{C}(y)} 0.$$

It was introduced and studied in [2]. Some existence results have been proved in [2] by using KKM-Fan Theorem [14].

When $\theta(y,v) = v$ and $C(y) = -\tilde{C}(y)$, the GVVLI becomes the problem of finding $y \in K$ such that

$$\forall x \in K, \ \exists v \in T(y) \ : \ \langle v, \eta(x,y) \rangle \not\geq_{-\mathrm{int}\, \tilde{C}(y)} 0.$$

It was investigated in [1, 4, 25].

When $\theta(y,v) = v$, $\eta(x,y) = x - y$ and $C(y) = -\tilde{C}(y)$, the GVVLI is equivalent to the problem of finding $y \in K$ such that

$$\forall x \in K, \ \exists v \in T(y) \ : \ \langle v, x - y \rangle \not\geq_{-\mathrm{int}\, \tilde{C}(y)} 0.$$

This problem was considered by Daniilidis and Hadjisavvas [13], Konnov and Yao [18], Lee et al [21, 23-24] and Lin et al [27].

When $Y = \mathbb{R}$, $L(X,Y) = X^*$, $C(x) = \mathbb{R}_-$, $\forall x \in K$, the GVVLI reduces to the following problem considered by Siddiqi et al [31] in the setting of reflexive Banach spaces: find $y \in K$ such that

$$\forall x \in K, \ \exists v \in T(y) \ : \ \langle \theta(y,v), \eta(x,y) \rangle \geq 0.$$

This is the weak formulation of the Generalized Variational–Like Inequality considered by Parida and Sen [28] and Yao [37-38].

It is clear that our GVVLI is more general and unifying one, which is one of the main motivations of this paper.

2. PRELIMINARIES

Let X and Y be topological vector spaces and X^* be the topological dual of X. Let $\langle L(X,Y), X\rangle$ be a dual system of $L(X,Y)$ and X.

Let $C : K \rightrightarrows Y$ be a multivalued map such that $\forall x \in K$, $C(x)$ is a proper, closed and convex cone with apex at the origin and with int $C(x) \neq \emptyset$. The following notations will be used in the sequel:

$$C_- := \bigcap_{x\in K} C(x) \quad \text{and} \quad C_+ := conv\{C(x) : x \in K\},$$

where $convA$ denotes the convex hull of the set A.

Definition 1. Let K and D be, respectively, nonempty subsets of X and X^*, and let P be a convex cone in Y. Given two maps $\theta : K \times D \to L(X,Y)$ and $\eta : K \times K \to X$, then a multivalued map $T : K \rightrightarrows D$ is called:

(i) $(P) - \eta$*-monotone with respect to* θ iff for every pair of points $x, z \in K$ and $\forall u \in T(x), \forall w \in T(z)$, we have

$$\langle \theta(x,u) - \theta(z,w), \eta(x,z)\rangle \in -P;$$

(ii) $(P) - \eta$*-pseudomonotone with respect to* θ iff for every pair of points $x, z \in K$ and $\forall u \in T(x), \forall w \in T(z)$, we have

$$\langle \theta(z,w), \eta(x,z)\rangle \in -P \quad \Rightarrow \quad \langle \theta(x,u), \eta(x,z)\rangle \in -P;$$

(iii) $C_z - \eta$*-pseudomonotone with respect to* θ iff for every pair of points $x, z \in K$ and $\forall u \in T(x)$, *for all* $w \in T(z)$, we have that

$$\langle \theta(z,w), \eta(x,z)\rangle \not\geq_{\text{int}\,(z)} 0 \quad \Rightarrow \quad \langle \theta(x,u), \eta(x,z)\rangle \not\geq_{\text{int}\,(z)} 0;$$

(iv) *weakly* $(P) - \eta$*-monotone with respect to* θ iff for every pair of points $x, z \in K$ and $\forall w \in T(z)$, $\exists u \in T(x)$, such that

$$\langle \theta(x,u) - \theta(z,w), \eta(x,z)\rangle \in -P;$$

(v) *weakly* $(P) - \eta$*-pseudomonotone with respect to* θ iff for every pair of points $x, z \in K$ and $\forall w \in T(z)$, we have

$$\langle \theta(z,w), \eta(x,z) \rangle \in -P \quad \Rightarrow \quad \langle \theta(x,u), \eta(x,z) \rangle \in -P,$$

for some $u \in T(x)$;

(vi) *weakly* $C_z - \eta$*-pseudomonotone with respect to* θ iff for every pair of points $x, z \in K$ and $\forall w \in T(z)$, we have

$$\langle \theta(z,w), \eta(x,z) \rangle \not\leq_{\mathrm{int}\,(z)} 0 \quad \Rightarrow \quad \langle \theta(x,u), \eta(x,z) \rangle \not\leq_{\mathrm{int}\,(z)} 0,$$

for some $u \in T(x)$;

(vii) V*-hemicontinuous with respect to* θ iff K is convex and $\forall x, z \in K$, $\forall \alpha \in]0,1[$ and $\forall t_\alpha \in T(\alpha x + (1-\alpha)z)$, $\exists t \in T(z)$ such that $\forall \hat{z} \in X$, $\langle \theta(z, t_\alpha), \hat{z} \rangle$ converges to $\langle \theta(z,t), \hat{z} \rangle$ as $\alpha \downarrow 0$.

Remark 1. (a) Definition 1 can be regarded as an extension of Definition 2.1 in [18].
(b) If $Y = \mathbf{R}$, $L(X,Y) = X^*$ and $P = \mathbf{R}_-$, then the mappings in Definition 1 (ii) and (v) are called η-pseudomonotone with respect to θ and weakly η-pseudomonotone with respect to θ, respectively.
(c) It is clear that (i) implies (ii) and (iv), (ii) implies (v), (iii) implies (vi), and (iv) implies (v).
(d) It is also easy to see that if T is $(C_-) - \eta$-monotone (respectively, weakly $C_z - \eta$-monotone) with respect to θ, then it is $C_z - \eta$-pseuodomonotone (respectively, weakly $C_z - \eta$-pseudomonotone) with respect to θ.

Let $s \in Y^*$, where Y^* is the topological dual of Y. Consider the map $\theta_s : K \times D \to X^*$, defined by

$$\langle \theta_s(z, T(z)), x \rangle = \langle s, \langle \theta(z, T(z)), x \rangle \rangle \quad , \quad \forall x, z \in K.$$

Consider also

$$H(s) := \{x \in Y : \langle s, x \rangle \leq 0\}$$

and

$$C_+^* := \{l \in Y^* : \langle l, x \rangle \leq 0, \ \forall x \in C_+\}.$$

Proposition 1. Let $T : K \rightrightarrows D$ be $(H(s)) - \eta$-pseudomonotone (respectively, weakly $(H(s)) - \eta$-pseudomonotone) with respect to θ, for some $s \in Y^* \setminus \{0\}$. Then T is η-pseudomonotone (respectively, weakly η-pseudomonotone) with respect to θ_s.

Proof. $\forall x, z \in K$, assume that

$$\langle \theta_s(z, w), \eta(x, z) \rangle \geq 0 \ \text{ or } \ -\langle \theta_s(z, w), \eta(x, z) \rangle \leq 0, \quad \forall w \in T(z).$$

Then $\langle s, -\langle \theta(z, w), \eta(x, z) \rangle \rangle \leq 0$, $\forall w \in T(z)$ and $\langle \theta(z, w), \eta(x, z) \rangle \in -H(s)$. Since T is $(H(s)) - \eta$-pseudomonotone with respect to θ, we must have

$$\langle \theta(x, u), \eta(x, z) \rangle \in -H(s) \quad , \quad \forall u \in T(x).$$

Hence, $\forall u \in T(x)$,

$$-\langle \theta_s(x, u), \eta(x, z) \rangle \leq 0 \ \text{ or } \ \langle \theta_s(x, u), \eta(x, z) \rangle \geq 0.$$

So, T is η-pseudomonotone with respect to θ_s. Analogously, we can prove the other part. $\square$

Definition 2. Let $W : X \rightrightarrows Y$ be a multivalued map. The *graph of* W, denoted by $\mathcal{G}(W)$, is

$$\mathcal{G}(W) := \{(x, z) \in X \times Y : x \in X, \ z \in W(x)\}.$$

The *inverse* W^{-1} of W is the multivalued map from $\mathcal{R}(W)$, the range of W, to X defined by

$$x \in W^{-1}(z) \quad \Leftrightarrow \quad z \in W(x).$$

In other words, $W^{-1}(z) := \{x \in X : (x, z) \in \mathcal{G}(W)\}$.

We mention a result of Chowdhury and Tan [11] which is a generalized form of Fan-Browder fixed point theorem [5, 14].

Theorem 1. Let K be a nonempty and convex subset of a topological vector space X and $A, B : K \rightrightarrows K \bigcup \{\emptyset\}$ be two multivalued maps, such that

1^0 $\forall z \in K,\ A(z) \subset B(z)$;

2^0 $\forall z \in K,\ B(z)$ is convex;

3^0 $\forall z \in K,\ A^{-1}(z)$ is compactly open (i.e., $A^{-1}(z) \bigcap L$ is open in L for each nonempty and compact subset L of K);

4^0 there exist a nonempty, closed and compact subset E of K and $\bar{z} \in E$, such that $K \setminus E \subset B^{-1}(\bar{z})$;

5^0 $\forall z \in E,\ A(z) \neq \emptyset$.

Then $\exists z_0 \in K$, such that $z_0 \in B(z_0)$.

The following well-known result plays a crucial role in the proofs of results of Sect.4.

Theorem 2 (Kneser) [17]. Let K be a nonempty and convex subset of a vector space, and let E be a nonempty, compact and convex subset of a Hausdorff topological vector space. Suppose that the functional $f : K \times E \to \mathbb{R}$ is such that, for each fixed $z \in K,\ f(z, \cdot)$ is lower semicontinuous and convex and that, for each fixed $x \in E,\ f(\cdot, x)$ is concave. Then

$$\min_{x \in E} \sup_{z \in K} f(z, x) = \sup_{z \in K} \min_{x \in E} f(z, x).$$

3. EXISTENCE RESULTS

Throughout this paper, the bilinear form $\langle \cdot, \cdot \rangle$ is supposed to be continuous. First of all we establish a generalized linearization lemma as follows:

Lemma 1. Let X and Y be topological vector spaces, K be a nonempty and convex subset of X, and D be a nonempty subset of X^*. Let $C : K \rightrightarrows Y$ be a multivalued map such that $\forall x \in K,\ C(x)$ is a proper, closed and convex cone in Y with apex at the origin and with $\operatorname{int} C(x) \neq \emptyset$. Let $\theta : K \times D \to L(X, Y),\ \eta : K \times K \to X$ and $T : K \rightrightarrows D$. We consider the following problems:

(I) Find $y \in K$ such that $\forall x \in K$, $\exists v \in T(y)$: $\langle \theta(y,v), \eta(x,y) \rangle \not\leq_{\operatorname{int} C(y)} 0$;

(II) Find $y \in K$ such that $\forall x \in K$, $\exists u \in T(x)$: $\langle \theta(x,u), \eta(x,y) \rangle \not\leq_{\operatorname{int} C(y)} 0$;

(III) Find $y \in K$ such that $\forall u \in T(x)$: $\langle \theta(x,u), \eta(x,y) \rangle \not\leq_{\operatorname{int} C(y)} 0$, $\forall x \in K$.

Then,

(i) Problem (I) implies Problem (II) if T is weakly $C_y - \eta$-pseudomonotone with respect to θ and, moreover, implies Problem (III) if T is $C_y - \eta$-pseudomonotone with respect to θ;

(ii) Problem (II) implies Problem (I) if T is V-hemicontinuous and, $\eta(\cdot,\cdot)$ and $\theta(\cdot,\cdot)$ are affine in their first arguments such that $\eta(x,x) = 0$, $\forall x \in K$;

(iii) Problem (III) implies Problem (II).

Proof. (i) It follows from the weak $C_y - \eta$-pseuodomonotonicity with respect to θ and $C_y - \eta$-pseudomonotonicity with respect to θ of T, respectively.
(ii) Let $y \in K$ be a solution of (II). Then $\forall x \in K$, $\exists u \in T(x)$ such that

$$\langle \theta(x,u), \eta(x,y) \rangle \not\leq_{\operatorname{int} C(y)} 0.$$

Set $x_\alpha := \alpha x + (1-\alpha)y$, for $\alpha \in]0,1[$. Since K is convex, $x_\alpha \in K$. Then $\exists u_\alpha \in T(x_\alpha)$, such that

$$\langle \theta(x_\alpha, u_\alpha), \eta(x_\alpha, y) \rangle \not\leq_{\operatorname{int} C(y)} 0.$$

Since $\theta(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$ are affine in their first arguments and $\eta(x,x) = 0$, $\forall x \in K$, we have

$$\alpha^2 \langle \theta(x,u_\alpha), \eta(x,y) \rangle + \alpha(1-\alpha) \langle \theta(y,u_\alpha), \eta(x,y) \rangle \not\leq_{\operatorname{int} C(y)} 0,$$

or

$$\alpha^2 \langle \theta(x,u_\alpha), \eta(x,y) \rangle + \alpha(1-\alpha) \langle \theta(y,u_\alpha), \eta(x,y) \rangle \notin \operatorname{int} C(y).$$

Since $C(y)$ is a convex cone, we get

$$\alpha \langle \theta(x,u_\alpha), \eta(x,y) \rangle + (1-\alpha) \langle \theta(y,u_\alpha), \eta(x,y) \rangle \notin \operatorname{int} C(y).$$

Taking $\alpha \downarrow 0$ and by V-hemicontinuity with respect to θ of T, $\exists v \in T(y)$, such that

$$\langle \theta(y,v), \eta(x,y) \rangle \notin \text{int } C(y), \quad \text{i.e.,} \quad \langle \theta(y,v), \eta(x,y) \rangle \not\geq_{\text{int } C(y)} 0.$$

Hence y is a solution of (I).
(iii) is obvious. □

Now we are ready to prove some existence theorems for the GVVLI under certain pseudomonotonicity assumptions.

Theorem 3. Let X and Y be topological vector spaces and let X^* be the topological dual of X. Let K be a nonempty and convex subset of X and D be a nonempty subset of X^*. Assume that the following conditions are satisfied:

(i) $C : K \rightrightarrows Y$ is a multivalued map such that $\forall x \in K$, $C(x)$ is a proper, closed and convex cone in Y with apex at the origin and with int $C(x) \neq \emptyset$;

(ii) $W : K \rightrightarrows Y$ is a multivalued map, defined by $W(x) := Y \setminus \{\text{int } C(x)\}$ $\forall x \in K$, such that $\mathcal{G}(W)$ is closed;

(iii) $\theta : K \times D \to L(X,Y)$ is affine in the first argument;

(iv) $\eta : K \times K \to X$ is continuous in the second argument and affine in the first argument such that $\eta(x,x) = 0$, $\forall x \in K$;

(v) $T : K \rightrightarrows D$ is $C_z - \eta$-pseudomonotone and V-hemicontinuous with respect to θ;

(vi) there exist a nonempty, closed and compact subset E of K and $\bar{z} \in E$, such that $\forall z \in K \setminus E$, we have

$$\forall w \in T(z), \ \langle \theta(z,w), \eta(\bar{z},z) \rangle \geq_{\text{int } C(z)} 0,$$

where the inequality means that $\langle \theta(z,w), \eta(\bar{z},z) \rangle \in \text{int } C(z)$.

Then, there exists a solution $y \in E$ to the GVVLI.

Proof. Consider two multivalued maps $A, B : K \rightrightarrows K \bigcup\{\emptyset\}$, defined by

$$A(z) := \{x \in K \ : \ \exists u \in T(x),\ \langle \theta(x,u), \eta(x,z)\rangle \geq_{\operatorname{int} C(z)} 0\}$$

and

$$B(z) := \{x \in K \ : \ \forall w \in T(z),\ \langle \theta(z,w), \eta(x,z)\rangle \geq_{\operatorname{int} C(z)} 0\} \quad , \quad \forall z \in K.$$

The proof is divided into the following five steps:
(a) For each $z \in K$, $A(z) \subset B(z)$: Let $x \in A(z)$. Then $\exists \bar{u} \in T(x)$,

$$\langle \theta(x,\bar{u}), \eta(x,z)\rangle \geq_{\operatorname{int} C(z)} 0. \tag{3.1}$$

Assume to the contrary that $x \notin B(z)$. Then $\exists w \in T(z)$,

$$\langle \theta(z,w), \eta(x,z)\rangle \geq_{\operatorname{int} C(z)} 0.$$

By $C_z - \eta$-pseudomonotonicity with respect to θ of T, we have $\forall u \in T(x)$,

$$\langle \theta(x,u), \eta(x,z)\rangle \geq_{\operatorname{int} C(z)} 0,$$

which contradicts to (3.1).
(b) For each $z \in K$, $B(z)$ is convex: Let $x_1, x_2 \in B(z)$ and $\alpha,\ \beta \geq 0$ such that $\alpha + \beta = 1$. Then $\forall w \in T(z)$,

$$\langle \theta(z,w), \eta(x_1,z)\rangle >_{C(z)} 0, \quad \text{i.e.,} \quad \langle \theta(z,w), \eta(x_1,z)\rangle \in \operatorname{int} C(z) \tag{3.2}$$

and

$$\langle \theta(z,w), \eta(x_2,z)\rangle >_{C(z)} 0, \quad \text{i.e.,} \quad \langle \theta(z,w), \eta(x_2,z)\rangle \in \operatorname{int} C(z). \tag{3.3}$$

Multiplying (3.2) by α and (3.3) by β, and then add resulting terms, we get

$$\langle \theta(z,w), \alpha\eta(x_1,z)\rangle + \langle \theta(z,w), \beta\eta(x_2,z)\rangle \in \operatorname{int} C(z) + \operatorname{int} C(z) \subseteq \operatorname{int} C(z).$$

Since $\eta(\cdot,\cdot)$ is affine in the first argument, we have

$$\langle \theta(z,w), \eta(\alpha x_1 + \beta x_2, z)\rangle \geq_{\operatorname{int} C(z)} 0.$$

Hence $\alpha x_1 + \beta x_2 \in B(z)$, as desired.
(c) For each $x \in K$, $A^{-1}(x)$ is compactly open (i.e., $A^{-1}(x) \bigcap L$ *is open in* L for each nonempty and compact subset L *of* K): Let $Q = A^{-1}(x) \bigcap L = \{z \in L : x \in A(z)\}$ and $\{z_n\}$ be a net in Q^c, the complement of Q in L, convergent to $z \in L$. Since $z_n \in Q^c$, $\forall u \in T(x)$,

$$\langle \theta(x,u), \eta(x,z_n)\rangle \geq_{\operatorname{int} C(z)} 0, \quad \text{i.e.,} \quad \langle \theta(x,u), \eta(x,z_n)\rangle \notin \operatorname{int} C(z_n)$$

and hence

$$\langle \theta(x,u), \eta(x,z_n)\rangle \in W(z_n) = Y \setminus \{\operatorname{int} C(z_n)\}.$$

Since $\theta(x,u) \in L(X,Y)$, $\forall x \in K$, $u \in T(x)$ and $\eta(\cdot,\cdot)$ is continuous in the second argument, we achieve that the net $\{\langle \theta(x,u), \eta(x,z_n)\rangle\}$ converges to $\langle \theta(x,u), \eta(x,z)\rangle \in Y$. Thus

$$(z_n, \langle \theta(x,u), \eta(x,z_n)\rangle) \text{ converges to } (z, \langle \theta(x,u), \eta(x,z)\rangle) \in \mathcal{G}(\mathcal{W}),$$

because $\mathcal{G}(\mathcal{W})$ is closed. Therefore, $\forall u \in T(x)$,

$$\langle \theta(x,u), \eta(x,z)\rangle \in W(z) = Y \setminus \{\operatorname{int} C(z)\},$$

and hence $z \in Q^c$. Consequently $A^{-1}(x) \bigcap L$ is open in L.
(d) By the hypothesis (vi), there exist a nonempty, closed and compact subset E of K and $\bar{z} \in E$ such that $\forall z \in K \setminus E$, we have

$$\forall w \in T(z), \ \langle \theta(z,w), \eta(\bar{z},z)\rangle \geq_{\operatorname{int} C(z)} 0.$$

Then $K \setminus E \subset B^{-1}(\bar{z})$.
(e) B has no fixed point: Suppose that B has a fixed point, say $z_0 \in K$. Then $\forall w_0 \in T(z_0)$, $\langle \theta(z_0,w_0), \eta(z_0,z_0)\rangle \geq_{\operatorname{int} C(z)} 0$. Since $\eta(z_0,z_0) = 0$, we have

$$\langle \theta(z_0,w_0), \eta(z_0,z_0)\rangle = 0 \in \operatorname{int} C(z_0),$$

and thus int $C(z_0)$ is an absorbing set in Y, which contradicts the assumption that $C(z_0)$ is proper in Y. Hence B has no fixed point.

Since B has no fixed point, we reach to a conclusion that either A or B would not satisfy at least one of the assumptions of Theorem 1. As we have seen above that A and B satisfy all the assumptions of Theorem 1 except 5^0, that is, $\forall z \in E$, $A(z)$ is nonempty. Hence there must be an $y \in E$ such that $A(y) = \emptyset$, namely, $\forall x \in K$,

$$\exists u \in T(x), \ \langle \theta(x,u), \eta(x,y)\rangle \not\geq_{\operatorname{int} C(y)} 0.$$

From Lemma 1, we have $y \in E$ such that

$$\forall x \in K,\ \exists v \in T(y),\ \langle \theta(y,v), \eta(x,y)\rangle \not\geq_{\operatorname{int} C(y)} 0. \qquad \square$$

We now obtain an existence theorem for the GVVLI for weakly $C_z - \eta$-pseudomonotone maps with respect to θ under additional assumptions.

Theorem 4. Let X, Y, X^*, K, E, D, C, W and η be same as in Theorem 3. Assume that the condition (vi) of Theorem 3 and the following conditions are satisfied:

(i) $\theta : K \times D \to L(X,Y)$ is continuous in the second argument and affine in the first argument;

(ii) $T : K \rightrightarrows D$ is compact valued, weakly $C_z - \eta$-pseudomonotone and V-hemicontinuous with respect to θ.

Then, there exists a solution $y \in E$ to the GVVLI.

Proof. Consider two multivalued maps $A, B : K \rightrightarrows K \bigcup\{\emptyset\}$, defined by

$$A(z) := \{x \in K \ :\ \forall u \in T(x),\ \langle \theta(x,u), \eta(x,z)\rangle \geq_{\operatorname{int} C(z)} 0\}$$

and

$$B(z) := \{x \in K \ :\ \forall w \in T(z),\ \langle \theta(z,w), \eta(x,z)\rangle \geq_{\operatorname{int} C(z)} 0\} \quad , \quad \forall z \in K.$$

By using the same arguments as in the proof (a) of Theorem 3 and weakly $C_z - \eta$ -pseudomonotonicity with respect to θ of T, we see that $A(z) \subset B(z)$, $\forall z \in K$.

We have already seen in the proof of Theorem 3 that $\forall z \in K$, $B(z)$ is convex and the multivalued map B has no fixed point.

Now we will see that, for each $x \in K$, $A^{-1}(x)$ is compactly open (i.e., $A^{-1}(x) \bigcap L$ is open in L for each nonempty and compact subset L of K).

Let $Q = A^{-1}(x) \bigcap L = \{z \in L : x \in A(z)\}$ and $\{z_n\}$ be a net in Q^c, the complement of Q in L, convergent to $z \in L$. Since $z_n \in Q^c$, then for some $u_n \in T(x)$ we have

$$\langle \theta(x,u_n), \eta(x,z_n)\rangle \geq_{\operatorname{int} C(z_n)} 0, \ \text{ i.e., } \ \langle \theta(x,u_n), \eta(x,z_n)\rangle \notin \operatorname{int} C(z_n), \quad \forall n$$

and hence

$$\langle \theta(x,u_n), \eta(x,z_n)\rangle \in W(z_n) = Y \setminus \{\text{int } C(z_n)\} \quad , \quad \forall n.$$

Since T is compact valued, the net $\{u_n\}$ has a convergent subnet $\{u_{n_k}\}$. We will still denote this subnet by $\{u_n\}$. Let $\{u_n\}$ converge to some $u \in T(x)$. Since $\theta(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$ are both continuous in their second arguments and $\theta(x,u) \in L(X,Y)$, $\forall x \in K$, $u \in T(x)$, we have the net $\{\langle \theta(x,u_n), \eta(x,z_n)\rangle\}$ converges to $\langle \theta(x,u), \eta(x,z)\rangle \in Y$. Thus $(z_n, \langle \theta(x,u_n), \eta(x,z_n)\rangle)$ converges to $(z, \langle \theta(x,u), \eta(x,z)\rangle) \in \mathcal{G}(\mathcal{W})$, because $\mathcal{G}(\mathcal{W})$ is closed. Therefore, for some $u \in T(x)$,

$$\langle \theta(x,u), \eta(x,z)\rangle \in W(z) = Y \setminus \{\text{int } C(z)\},$$

and hence $z \in Q^c$. Consequently $A^{-1}(x) \bigcap L$ is open in L.

Hence, as in the proof of Theorem 3, there must be an $y \in E$ such that $A(y) = \emptyset$, namely, for each $x \in K$,

$$\forall u \in T(x), \ \langle \theta(x,u), \eta(x,y)\rangle \not\leq_{\text{int } C(y)} 0.$$

From Lemma 1, we have $y \in E$ such that

$$\forall z \in K, \ \exists v \in T(y), \ \langle \theta(y,v), \eta(x,y)\rangle \not\geq_{\text{int } C(y)} 0. \qquad \square$$

Next, we shall prove an existence result without any kind of pseudomonotonicity assumption.

Theorem 5. Let X, Y, X^*, K, E, C, W and η be same as in Theorem 3. Assume that the condition (vi) of Theorem 3 and the following conditions are satisfied:

(i) D is a compact subset of X^*;

(ii) $\theta : K \times D \to L(X,Y)$ is continuous in both the arguments and affine in the first argument;

(iii) $T : K \rightrightarrows D$ is a multivalued map such that its graph is closed.

Then there exists a solution $y \in E$ to the GVVLI.

Proof. Consider the multivalued map $B : K \rightrightarrows K \bigcup\{\emptyset\}$, defined by

$$B(z) = \{x \in K \ : \ \forall w \in T(z),\ \langle \theta(z,w), \eta(x,z)\rangle \geq_{\operatorname{int} C(z)} 0\} \quad , \quad \forall z \in K.$$

As we have seen in the proof of Theorem 3, that $\forall z \in K, B(z)$ is a convex subset of K and the multivalued map B has no fixed point.

Now we have only to show that, $\forall x \in K,\ B^{-1}(x)$ is compactly open (i.e., $B^{-1}(x) \bigcap L$ is open in L for each nonempty and compact subset L of K).

Let $Q = B^{-1}(x) \bigcap L = \{z \in L : x \in B(z)\}$ and $\{z_n\}$ be a net in Q^c, the complement of Q in L, convergent to $z \in L$. Since $z_n \in Q^c$, then for some $w_n \in T(z_n)$, we have

$$\langle \theta(z_n, w_n), \eta(x, z_n)\rangle \not\geq_{\operatorname{int} C(z)} 0, \quad \text{i.e.,} \quad \langle \theta(z_n, w_n), \eta(x, z_n)\rangle \notin \operatorname{int} C(z_n), \quad \forall n$$

and hence

$$\langle \theta(z_n, w_n), \eta(x, z_n)\rangle \in W(z_n) \quad , \quad \forall n.$$

Since $T(K)$ is contained in a compact set D, we may assume that w_n converges to some $w \in D$, then by the definition of closed graph of T, we have $w \in T(z)$. Since $\eta(\cdot,\cdot)$ in the second argument and $\theta(\cdot,\cdot)$ in both the arguments are continuous, and $\theta(x,u) \in L(X,Y),\ \forall x \in K,\ u \in T(x)$, we have the net $\{\langle \theta(z_n, w_n), \eta(x, z_n)\rangle\}$ converges to $\langle \theta(z,w), \eta(x,z)\rangle \in Y$. Thus $(z_n, \langle \theta(z_n, w_n), \eta(x, z_n)\rangle)$ converges to $(z, \langle \theta(z,w), \eta(x,z)\rangle) \in \mathcal{G}(\mathcal{W})$, because $\mathcal{G}(\mathcal{W})$ is closed. Therefore, for some $w \in T(z)$,

$$\langle \theta(z,w), \eta(x,z)\rangle \in W(x) = Y \setminus \{\operatorname{int} C(z)\},$$

and hence $z \in Q^c$. Consequently $B^{-1}(x) \bigcap L$ is open in L.

Hence, as in the proof of Theorem 3, there must be an $y \in E$, such that $B(y) = \emptyset$, i.e, there exists $y \in E$, such that

$$\forall x \in K,\ \exists v \in T(y),\ \langle \theta(y,v), \eta(x,y)\rangle \not\geq_{-\operatorname{int} C(z)} 0. \qquad \square$$

Remark 2. We note that in all results of this section, we neither assumed X and Y are Hausdorff nor $\forall x \in K,\ C(x)$ is pointed.

4. SCALARIZATION

In this section, we use the technique of Konnov and Yao [18] to derive some existence theorems for the GVVLI by way of solving an appropriate Generalized Variational–Like Inequality (for short, GVLI).

Through out in this section we assume that X is a Hausdorff topological vector space.

Theorem 6. Let X, Y, X^*, K, D, W, η and θ be same as in Theorem 3. Assume that the following conditions are satisfied:

(i) $C : K \rightrightarrows Y$ is defined as in Theorem 3 such that $C_+^* \setminus \{0\} \neq \emptyset$;

(ii) $T : K \rightrightarrows D$ is V-hemicontinuous and $(H(s)) - \eta$-pseudomonotone with respect to θ for some $s \in C_+^* \setminus \{0\}$, where $H(s) \neq Y$.

(iii) there exist a nonempty and compact subset E of K and $\bar{z} \in E$, such that $\forall z \in K \setminus E$, we have

$$\forall w \in T(z),\ \langle \theta(z,w), \eta(\bar{z},z)\rangle \geq_{\operatorname{int} C(z)} 0.$$

Then, there exists a solution $y \in E$ to the GVVLI.
If, in addition,

(iv) $\forall z \in K$, $T(z)$ is convex and compact;

(v) $\forall x, z \in K$, $w \mapsto \langle \theta(z,w), \eta(x,z)\rangle$ is lower semicontinuous and convex;

(vi) $\forall x \in K$, $z \mapsto \langle \theta(z,w), \eta(x,z)\rangle$ is concave.

Then, there exists a strong solution $y \in E$ to the GVVLI.

Proof. (a) Since $H(s) \neq Y$, it can be shown that int $H(s) = s^{-1}(]0, -\infty[)$ (see, [23]). As $s \in C_+^* \setminus \{0\}$, the multivalued map T is η-pseudomonotone with respect to θ_s due to Proposition 1. Now, in the special case where $Y = \mathbf{R}$, $C(z) = \mathbf{R}_-$, $\forall z \in K$, Theorem 3 guarantees the existence of a solution $y \in E$ to the GVLI$_s$, i.e., $\exists y \in E$ such that

$$\forall x \in K,\ \exists v \in T(y),\ \langle \theta_s(y,v), \eta(x,y)\rangle \geq 0. \tag{4.1}$$

Consequently, $\forall x \in K$, $\exists v \in T(y)$, such that

$$\langle s, \langle \theta(y,v), \eta(x,y)\rangle\rangle \geq 0,$$

and hence $\langle \theta(y,v), \eta(x,y)\rangle \notin \text{int } H(s)$. Since $s \in C_+^*$, $\text{int } H(s) \supseteq \text{int } C_+ \supseteq \text{int } C(y)$, so that

$$\langle \theta(y,v), \eta(x,y)\rangle \notin \text{int } C(y).$$

(b) We define a real-valued function $f : K \times T(y) \to \mathbf{R}$ by

$$f(z,v) = \langle \theta_s(y,v), \eta(x,y)\rangle.$$

By definition, for $s \in Y^*$, we have

$$\langle \theta_s(y,v), \eta(x,y)\rangle = \langle s, \langle \theta(y,v), \eta(x,y)\rangle\rangle.$$

From condition (iv) and (v), we have $\forall x \in K$, $f(x,\cdot)$, is lower semicontinuous and convex, and $\forall v \in T(y)$, $f(\cdot,v)$, is concave. Then from (4.1) and Theorem 2, we have

$$\max_{v \in T(y)} \min_{x \in K} \langle \theta_s(y,v), \eta(x,y)\rangle = \min_{x \in K} \max_{v \in T(y)} \langle \theta_s(y,v), \eta(x,y)\rangle \geq 0.$$

Since $T(y)$ is compact, $\exists v \in T(y)$ such that

$$\langle \theta_s(y,v), \eta(x,y)\rangle \geq 0, \quad \forall x \in K,$$

i.e., $\exists v \in T(y)$ such that

$$\langle s, \langle \theta(y,v), \eta(x,y)\rangle\rangle \geq 0, \quad \forall x \in K.$$

Analogously, it follows that

$$\langle \theta(y,v), \eta(x,y)\rangle \notin \text{int } C(y), \quad \text{i.e.,} \quad \langle \theta(y,v), \eta(x,y)\rangle \not\geq_{\text{int } C(y)} 0 \quad , \quad \forall x \in K.$$

Therefore, $y \in E$ is a strong solution of the GVVLI. □

Theorem 7. Let X, Y, X^*, K, E, D, C, W and η be same as in Theorem 6. Assume that the condition (iii) of Theorem 6 and the following conditions are satisfied:

(i) $\theta : K \times D \to L(X,Y)$ is continuous in the second argument and affine in the first argument;

(ii) $T : K \rightrightarrows D$ is compact–valued, V–hemicontinuous and weakly $H(s) - \eta$–pseudomonotone with respect to θ for some $s \in C_+^* \setminus \{0\}$, where $H(s) \neq Y$.

Then there exists a solution $y \in E$ to the GVVLI.
If, in addition,

(iii) $\forall z \in K$, $T(z)$ is convex;

(iv) $\theta(\cdot,\cdot)$ is convex in the second argument;

(v) $\forall x \in K$, $z \mapsto \langle \theta(z,w), \eta(x,z) \rangle$ is concave.

Then, there exists a strong solution $y \in E$ to the GVVLI.

Proof. It follows from Theorem 4 that there is a solution $y \in E$ of the GVVLI. By the same arguments as that in Theorem 6, we can show that y is a strong solution of the GVVLI. □

Theorem 8. Let X, Y, X^*, K, E, C, W, η be same as in Theorem 6. Assume that the condition (iii) of Theorem 6 and the following conditions are satisfied:

(i) D is a compact subset in X^*;

(ii) $\theta : K \times D \to L(X,Y)$ is continuous in both the arguments and affine in the first argument;

(iii) $T : K \rightrightarrows D$ is a multivalued map such that its graph is closed.

Then, there exists a solution $y \in E$ to the GVVLI.
If, in addition, conditions (iii), (iv) and (v) of Theorem 7 hold, then there exists a strong solution $y \in E$ to the GVVLI.

Proof. It follows from Theorem 5 that there is $y \in E$ which is a solution of the GVVLI. Again by the same arguments as that in Theorem 6, we see that y is a strong solution of the GVVLI. □

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