

## ON GENERALISED MIXED CO-QUASI-VARIATIONAL INEQUALITIES WITH NONCOMPACT VALUED MAPPINGS

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In this paper, we consider generalised mixed co-quasi-variational inequalities with noncompact valued mappings and propose an iterative algorithm for computing their approximate solutions. We prove that the approximate solutions obtained by the proposed algorithm converge to the exact solution of our co-quasi-variational inequality. Some special cases are also discussed.

### 1. INTRODUCTION AND FORMULATION

The projection iterative method is one of the most important and useful methods for finding the approximate solutions of fixed point problems, and variational and quasi-variational inequality problems; See for example [5, 6, 8, 9, 10, 11, 13, 14] and references therein. In most of the papers appearing in the literature on this topic, the metric projection operators, as in Hilbert spaces, are used. But it is impossible to use the metric projection operator in the setting of Banach spaces because these operators are not nonexpansive. Recently, Takahashi and Kim [15] used the sunny nonexpansive retraction to set up an iterative scheme for finding a fixed point of a nonexpansive and nonself mapping in Banach spaces. Inspired by the work of Takahashi and Kim [15], Alber and Yao [2] used a sunny nonexpansive retraction to construct the projection iterative method for finding the approximate solutions of a class of multivalued quasi-variational inequalities in Banach spaces. They gave the name *co-quasi-variational inequality* for a quasi-variational inequality in Banach spaces and presented an iterative algorithm. They also proved several convergence results for approximate solutions obtained by their algorithms and in particular several existence results were obtained. Recently, Chang [4] also studied the existence and convergence of solutions of the Mann and Ishikawa iterative processes for a class of variational inclusions with accretive type mappings in Banach spaces. The mathematical approach in [4] is quite different from the one used by Alber and Yao [2].

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Let  $B$  be a real Banach space with norm  $\|\cdot\|$ ,  $B^*$  its topological conjugate space with norm  $\|\cdot\|_*$  and  $\langle x, f \rangle$  be a pairing between  $x \in B$  and  $f \in B^*$ . Given single-valued mappings  $f, g, p, G : B \rightarrow B$  and multivalued mappings  $M, S, T, K : B \rightarrow 2^B$  such that  $\forall x \in B$ ,  $K(x)$  is a nonempty, closed and convex set, we consider the following generalised mixed co-quasi-variational inequality problem :

$$\left\{ \begin{array}{l} \text{Find } x \in B, u \in M(x), v \in S(x) \text{ and } w \in T(x) \\ \text{such that } G(x) \in K(x) \text{ and} \\ \langle p(u) - (f(v) - g(w)), J(z - G(x)) \rangle \geq 0, \quad \forall z \in K(x), \end{array} \right.$$

where  $J : B \rightarrow B^*$  is the normalised duality mapping and  $2^B$  is the family of all nonempty subsets of  $B$ .

The nonlinear operator  $J : B \rightarrow B^*$  is called *normalised duality mapping* if

$$\|Jx\|_* = \|x\| \text{ and } \langle x, J(x) \rangle = \|x\|^2, \quad \forall x \in B.$$

For further detail of the duality mapping  $J$ , we refer to [1].

Let us see some special cases of generalised mixed co-quasi-variational inequality problems.

#### SPECIAL CASES:

- (i) If  $p \equiv 0$ ,  $g$  and  $M$  are identity mappings and  $T$  is single-valued mapping, then generalised mixed co-quasi-variational inequality problem reduces to the problem of finding  $x \in B$  and  $v \in S(x)$  such that  $G(x) \in K(x)$  and

$$(1.1) \quad \langle T(x) - f(v), J(z - G(x)) \rangle \geq 0, \quad \forall z \in K(x),$$

A problem similar to (1.1) is recently considered and studied by Alber and Yao [2].

- (ii) If  $B$  is a Hilbert space,  $f, g$  and  $M$  are identity mappings,  $G(x) = p(x)$  and  $K(x) = K \forall x \in B$  then generalised mixed co-quasi-variational inequality problem becomes the following generalised variational inequality problem considered and studied by Verma [16]:

$$\left\{ \begin{array}{l} \text{Find } x \in B, v \in S(x) \text{ and } w \in T(x) \text{ such that} \\ \langle p(x) - (v - w), z - p(x) \rangle \geq 0, \quad \forall z \in K. \end{array} \right.$$

It is clear that the generalised mixed co-quasi-variational inequality problem includes many kinds of quasi-variational inequality problems, variational inequality problems and complementarity problems as special cases, such as [8, 13, 16, 17].

In this paper, we extend the approach of Alber and Yao [2] to a more general and unified problem called the generalised mixed co-quasi-variational inequality problem. We suggest and analyse an iterative algorithm to compute the approximate solutions of the generalised mixed co-quasi-variational inequality problem with noncompact valued mappings. We also prove convergence result for the approximate solutions obtained by the proposed algorithm.

## 2. PRELIMINARIES

We recall that the *uniform convexity* of the Banach space  $B$  means that for any given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\forall x, y \in B$ ,  $\|x\| \leq 1$ ,  $\|y\| \leq 1$ ,  $\|x - y\| = \varepsilon$  ensure the following inequality

$$\|x + y\| \leq 2(1 - \delta).$$

The function

$$\delta_B(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = 1, \|y\| = 1, \|x - y\| = \varepsilon \right\}$$

is called the *modulus of the convexity* of the space  $B$ .

The *uniform smoothness* of the space  $B$  means that for any given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \leq \varepsilon \|y\|$$

holds. The function

$$\rho_B(t) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t \right\}$$

is called the *modulus of the smoothness* of the space  $B$ .

We remark that the space  $B$  is uniformly convex if and only if  $\delta_B(\varepsilon) > 0$  for all  $\varepsilon > 0$ , and it is uniformly smooth if and only if  $\lim_{t \rightarrow 0} t^{-1} \rho_B(t) = 0$ . The following inequalities will be used in the proof of our main result.

**PROPOSITION 2.1.** ([1]) *Let  $B$  be a uniformly smooth Banach space and  $J$  be a normalised duality mapping from  $B$  to  $B^*$ . Then,  $\forall x, y \in B$ , we have*

$$(i) \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle,$$

$$(ii) \quad \langle x - y, J(x) - J(y) \rangle \leq 2d^2 \rho_B(4\|x - y\|/d), \text{ where } d = \sqrt{(\|x\|^2 + \|y\|^2)/2}.$$

We next recall the following definitions.

**DEFINITION 2.1:** A mapping  $A : B \rightarrow B$  is said to be

- (i) *strongly accretive* if there exists a constant  $\gamma > 0$  such that

$$\langle A(x) - A(y), J(x - y) \rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in B;$$

(ii) *Lipschitz continuous* if there exists a positive constant  $\beta$  such that

$$\|A(x) - A(y)\| \leq \beta \|x - y\|, \forall x, y \in B.$$

**DEFINITION 2.2:** A multivalued map  $S : B \rightarrow CB(B)$  is said to be  $\mathcal{H}$ -Lipschitz continuous if there exists a constant  $h > 0$  such that

$$\mathcal{H}(S(x), S(y)) \leq h \|x - y\|, \forall x, y \in B.$$

where  $\mathcal{H}(\cdot, \cdot)$  is the Hausdorff metric and  $CB(B)$  is the family of all nonempty closed bounded subsets of  $B$ .

**DEFINITION 2.3:** ([3, 7]) Let  $B$  be a real Banach space and  $\Omega$  a nonempty closed convex subset of  $B$ . A mapping  $Q_\Omega : B \rightarrow \Omega$  is said to be

- (i) *retraction* on  $\Omega$  if  $Q_\Omega^2 = Q_\Omega$ ;
- (ii) *nonexpansive retraction* if in addition

$$\|Q_\Omega x - Q_\Omega y\| \leq \|x - y\|, \forall x, y \in B;$$

- (iii) *sunny retraction* if  $\forall x \in B$ ,

$$Q_\Omega(Q_\Omega x + t(x - Q_\Omega x)) = Q_\Omega x, \forall t \in \mathbb{R}.$$

The following characterisation of a sunny nonexpansive retraction mapping can be found in [3, 7].

**PROPOSITION 2.2.**  $Q_\Omega$  is a sunny nonexpansive retraction if and only if  $\forall x \in B$  and  $\forall y \in \Omega$

$$\langle x - Q_\Omega x, J(Q_\Omega x - y) \rangle \geq 0.$$

From the above Proposition 2.2, we have the following retraction shift equality.

**PROPOSITION 2.3.** Let  $B$  be a Banach space,  $\Omega$  a nonempty closed and convex subset of  $B$  and  $m : B \rightarrow B$  a mapping. Then  $\forall x \in B$ , we have

$$Q_{\Omega+m(x)}x = m(x) + Q_\Omega(x - m(x)).$$

### 3. AN ITERATIVE ALGORITHM

In this section, we first derive some characterisations of solutions of generalised mixed co-quasi-variational inequality problem.

We mention the following characterisation theorem for the solution of generalised mixed co-quasi-variational inequality problem which can be easily proved by using Proposition 2.2 and the argument of [8, Theorem 3.1] and [1, Theorem 8.1].

**THEOREM 3.1.** Let  $B$  be a Banach space,  $f, g, p, G : B \rightarrow B$  single-valued mappings,  $M, S, T : B \rightarrow CB(B)$  mappings and  $K : B \rightarrow 2^B$  multivalued mappings such that  $\forall x \in B$ ,  $K(x)$  is a nonempty, closed and convex subset. Then the following statements are equivalent:

- (i) The set of elements  $(x, u, v, w)$  such that  $x \in B$ ,  $u \in M(x)$ ,  $v \in S(x)$  and  $w \in T(x)$  is a solution of generalised mixed co-quasi-variational inequality problem.
- (ii)  $x \in B$ ,  $u \in M(x)$ ,  $v \in S(x)$ ,  $w \in T(x)$  and

$$G(x) = Q_{K(x)} \left[ G(x) - \tau \left( p(u) - (f(v) - g(w)) \right) \right] \text{ for any } \tau > 0.$$

Combining Proposition 2.3 and Theorem 3.1, we obtain the following result.

**THEOREM 3.2.** Let  $B$  be a real Banach space,  $X$  a nonempty, closed and convex subset of  $B$ . Let  $f, g, p, m, G : B \rightarrow B$  be single-valued mappings, and  $M, S, T : B \rightarrow CB(B)$  mappings and  $K : B \rightarrow 2^B$  multivalued mappings such that  $\forall x \in B$ ,  $K(x) = m(x) + X$ . Then the set of elements  $(x, u, v, w)$  such that  $x \in B$ ,  $u \in M(x)$ ,  $v \in S(x)$  and  $w \in T(x)$  is a solution of generalised mixed co-quasi-variational inequality problem if and only if

$$x = x - G(x) + m(x) + Q_X \left[ G(x) - \tau \left( p(u) - (f(v) - g(w)) \right) - m(x) \right], \text{ for any } \tau > 0.$$

To compute the approximate solutions of generalised mixed co-quasi-variational inequality problem, we propose the following iterative algorithm.

**ALGORITHM 3.1.** Let  $K(x) = m(x) + X$ , where  $X$  is a nonempty, closed and convex subset of  $B$  and  $\tau > 0$  be fixed. Let  $f, g, p, m, G : B \rightarrow B$  be single-valued mappings and  $S, M, T : B \rightarrow CB(B)$  be multivalued mappings. For given  $u_0 \in M(x_0)$ ,  $v_0 \in S(x_0)$  and  $w_0 \in T(x_0)$ , let

$$x_1 = x_0 - G(x_0) + m(x_0) + Q_X \left[ G(x_0) - \tau \left( p(u_0) - (f(v_0) - g(w_0)) \right) - m(x_0) \right].$$

Since  $u_0 \in M(x_0) \in CB(B)$ ,  $v_0 \in S(x_0) \in CB(B)$ ,  $w_0 \in T(x_0) \in CB(B)$ , by Nadler [12], there exist  $u_1 \in M(x_1)$ ,  $v_1 \in S(x_1)$ ,  $w_1 \in T(x_1)$  such that

$$\begin{aligned} \|u_0 - u_1\| &\leq (1 + 1)\mathcal{H}(M(x_0), M(x_1)), \\ \|v_0 - v_1\| &\leq (1 + 1)\mathcal{H}(S(x_0), S(x_1)), \\ \|w_0 - w_1\| &\leq (1 + 1)\mathcal{H}(T(x_0), T(x_1)). \end{aligned}$$

Let

$$x_2 = x_1 - G(x_1) + m(x_1) + Q_X \left[ G(x_1) - \tau \left( p(u_1) - (f(v_1) - g(w_1)) \right) - m(x_1) \right].$$

By induction, we can obtain the sequences  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{v_n\}$  and  $\{w_n\}$  as

$$(3.1) \quad x_{n+1} = x_n - G(x_n) + m(x_n) + Q_X \left[ G(x_n) - \tau \left( p(u_n) - (f(v_n) - g(w_n)) \right) - m(x_n) \right],$$

$$\begin{aligned} u_n &\in M(x_n), \quad \|u_n - u_{n+1}\| \leq (1 + (n+1)^{-1}) \mathcal{H}(M(x_n), M(x_{n+1})), \\ v_n &\in S(x_n), \quad \|v_n - v_{n+1}\| \leq (1 + (n+1)^{-1}) \mathcal{H}(S(x_n), S(x_{n+1})), \\ w_n &\in T(x_n), \quad \|w_n - w_{n+1}\| \leq (1 + (n+1)^{-1}) \mathcal{H}(T(x_n), T(x_{n+1})), \end{aligned}$$

$n = 0, 1, 2, 3, \dots$

Finally, we prove that the approximate solutions obtained by Algorithm 3.1 converge to the exact solution of generalised mixed co-quasi-variational inequality problem. We also prove the existence of a solution of generalised mixed co-quasi-variational inequality problem.

**THEOREM 3.3.** *Let  $B$  be a real uniformly smooth Banach space with the module of smoothness  $\tau_B(t) \leq Dt^2$  for some  $D > 0$ . Let  $X$  be a nonempty, closed and convex subset of  $B$ ,  $f, g, p, m, G : B \rightarrow B$  single-valued mappings and  $M, S, T : B \rightarrow CB(B)$  and  $K : B \rightarrow 2^B$  multivalued mappings such that  $\forall x \in B$ ,  $K(x) = m(x) + X$ . Suppose that the following conditions are satisfied:*

- (i)  $f, g$  and  $p$  are Lipschitz continuous with corresponding constants  $\xi, \tau$  and  $\sigma$ , respectively.
- (ii)  $G$  is both strongly accretive with constant  $\gamma$  and Lipschitz continuous with constant  $\delta$ .
- (iii)  $M, S, T$  are  $\mathcal{H}$ -Lipschitz continuous with corresponding constants  $s, h$  and  $d$ , respectively.
- (iv)  $m$  is Lipschitz continuous with constant  $\theta$ .
- (v)  $0 < 2(1 - 2\gamma + 64D\delta^2)^{1/2} + 2\theta + \tau\sigma s + [1 + \tau(\xi h - rd)] < 1$ .

Then there exists a set of elements  $(x, u, v, w)$  such that  $x \in B$ ,  $u \in M(x)$ ,  $v \in S(x)$  and  $w \in T(x)$  which is a solution of generalised mixed co-quasi-variational inequality problem and  $x_n \rightarrow x$ ,  $u_n \rightarrow u$ ,  $v_n \rightarrow v$ ,  $w_n \rightarrow w$  as  $n \rightarrow \infty$ , where  $\{x_n\}$ ,  $\{u_n\}$ ,  $\{v_n\}$  and  $\{w_n\}$  are the sequences obtained by Algorithm (3.1).

**PROOF:** By the iterative scheme 3.1 and Proposition 2.3, we have

$$\begin{aligned} (3.2) \quad \|x_{n+1} - x_n\| &= \left\| x_n - G(x_n) + m(x_n) + Q_X \left[ G(x_n) - \tau(p(u_n) - (f(v_n) \right. \right. \\ &\quad \left. \left. - g(w_n))) - m(x_n) \right] - (x_{n-1} - G(x_{n-1})) + m(x_{n-1}) \right. \\ &\quad \left. - Q_X \left[ G(x_{n-1}) - \tau(p(u_{n-1}) - (f(v_{n-1}) - g(w_{n-1}))) - m(x_{n-1}) \right] \right\| \\ &\leq \left\| x_n - x_{n-1} - (G(x_n) - G(x_{n-1})) \right\| + 2\|m(x_n) - m(x_{n-1})\| \\ &\quad + \left\| x_n - x_{n-1} - (G(x_n) - G(x_{n-1})) \right\| + \tau\|p(u_n) - p(u_{n-1})\| \\ &\quad + \left\| x_n - x_{n-1} + \tau(f(v_n) - f(v_{n-1})) - \tau(g(w_n) - g(w_{n-1})) \right\| \\ &= 2\left\| x_n - x_{n-1} - (G(x_n) - G(x_{n-1})) \right\| + 2\|m(x_n) - m(x_{n-1})\| \end{aligned}$$

$$+\tau\|p(u_n) - p(u_{n-1})\| + \|x_n - x_{n-1} + \tau(f(v_n) - f(v_{n-1})) - \tau(g(w_n) - g(w_{n-1}))\|$$

By Proposition 2.1, we have (see, for example, the proof of [2, Theorem 3])

$$(3.3) \quad \|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\|^2 \leq (1 - 2\gamma + 64D\delta^2)\|x_n - x_{n-1}\|^2.$$

It is clear that

$$(3.4) \quad \|m(x_n) - m(x_{n-1})\| \leq \theta\|x_n - x_{n-1}\|.$$

Since  $M, S$  and  $T$  are  $\mathcal{H}$ -Lipschitz continuous, and  $f, g$  and  $p$  are Lipschitz continuous, we have

$$(3.5) \quad \|p(u_n) - p(u_{n-1})\| \leq \sigma\|u_n - u_{n-1}\| \leq \sigma s\left(1 + \frac{1}{n}\right)\|x_n - x_{n-1}\|,$$

$$(3.6) \quad \|f(v_n) - f(v_{n-1})\| \leq \xi\|v_n - v_{n-1}\| \leq \xi h\left(1 + \frac{1}{n}\right)\|x_n - x_{n-1}\|,$$

$$(3.7) \quad \|g(w_n) - g(w_{n-1})\| \leq r\|w_n - w_{n-1}\| \leq rd\left(1 + \frac{1}{n}\right)\|x_n - x_{n-1}\|.$$

From (3.6)–(3.7), it follows that

$$(3.8) \quad \begin{aligned} & \|x_n - x_{n-1} + \tau(f(v_n) - f(v_{n-1})) - \tau(g(w_n) - g(w_{n-1}))\| \\ & \leq \|x_n - x_{n-1}\| + \tau\|f(v_n) - f(v_{n-1})\| - \tau\|g(w_n) - g(w_{n-1})\| \\ & \leq \|x_n - x_{n-1}\| + \tau\xi h\left(1 + \frac{1}{n}\right)\|x_n - x_{n-1}\| - \tau rd\left(1 + \frac{1}{n}\right)\|x_n - x_{n-1}\| \\ & = [1 + \tau(1 + 1/n)(\xi h - rd)]\|x_n - x_{n-1}\|. \end{aligned}$$

Combining (3.2)–(3.5) and (3.8), it follows that

$$\|x_{n+1} - x_n\| \leq t_n\|x_n - x_{n-1}\|,$$

where  $t_n = 2(1 - 2\gamma + 64D\delta^2)^{1/2} + 2\theta + \tau\sigma s(1 + (1/n)) + [1 + \tau(1 + 1/n)(\xi h - rd)]$ .

Let  $t = 2(1 - 2\gamma + 64D\delta^2)^{1/2} + 2\theta + \tau\sigma s + [1 + \tau(\xi h - rd)]$ . Then  $t_n \rightarrow t$  as  $n \rightarrow \infty$ . It follows from (v) that  $t < 1$ . Hence  $t_n < 1$  for  $n$  sufficiently large. Consequently  $\{x_n\}$  is a Cauchy sequence in  $B$ . Since  $B$  is complete, we can let  $x_n \rightarrow x \in B$ . Now we prove that  $u_n \rightarrow u \in M(x)$ ,  $v_n \rightarrow v \in S(x)$  and  $w_n \rightarrow w \in T(x)$ . In fact, it follows from Algorithm 3.1 that

$$\begin{aligned} \|u_n - u_{n-1}\| & \leq \left(1 + \frac{1}{n}\right)s\|x_n - x_{n-1}\|, \\ \|v_n - v_{n-1}\| & \leq \left(1 + \frac{1}{n}\right)h\|x_n - x_{n-1}\|, \\ \|w_n - w_{n-1}\| & \leq \left(1 + \frac{1}{n}\right)d\|x_n - x_{n-1}\|, \end{aligned}$$

which implies that  $\{u_n\}$ ,  $\{v_n\}$  and  $\{w_n\}$  are also Cauchy sequences in  $B$ . Again, since  $B$  is complete, we can let  $u_n \rightarrow u$ ,  $v_n \rightarrow v$ ,  $w_n \rightarrow w$  as  $n \rightarrow \infty$ . Since  $Q_X, G, f, g, p, M, S, T$  and  $m$  are continuous in  $B$ , we have

$$x = x - G(x) + m(x) + Q_X \left[ G(x) - \tau \left( p(u) - (f(v) - g(w)) \right) - m(x) \right].$$

Further, we have

$$\begin{aligned} d(v, S(x)) &= \inf \{ \|v - y\| : y \in S(x) \} \\ &\leq \|v - v_n\| + d(v_n, S(x)) \\ &\leq \|v - v_n\| + \mathcal{H}(S(x_n), S(x)) \\ &\leq \|v - v_n\| + h\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence  $v \in S(x)$ . Similarly we can prove that  $u \in M(x)$ ,  $w \in T(x)$ . The result then follows from Theorem 3.2.  $\square$

REMARK 3.1. As a special case of Algorithm 3.1, we can easily derive an iterative algorithm, similar to the algorithm proposed in [2], for finding the approximation solutions of problem (1.1).

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