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GENERALISED VARIATIONAL-LIKE INEQUALITIES AND A GAP FUNCTION

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In this paper, we study the existence of solutions of generalised variational-like inequality problems by using a generalised form of the Fan-KKM-Theorem. We also introduce a gap function for generalised variational-like inequalities.

1. INTRODUCTION AND PRELIMINARIES

Let E be a topological vector space with dual E^* and let $\langle E^*, E\rangle$ be the dual system of E^* and E. We denote by 2^X the family of all nonempty subsets of a set X and by $\mathcal{F}(X)$ the family of all nonempty finite subsets of X. If X is a subset of a topological vector space E, we shall denote by $\overline{X}$ the closure of X in E, and by $\operatorname{co}(X)$ the convex hull of X. Let C and K be nonempty subsets of E and E^*, respectively. Given two maps $\theta : C \times K \to E^*$ and $\eta : C \times C \to E$, and a multifunction $T : C \to 2^K$, then we consider the following *generalised variational-like inequality problems*:

PROBLEM 1. Find $\overline{x} \in C$ and $\overline{s} \in T(\overline{x})$ such that

$$\langle \theta(\overline{x}, \overline{s}), \eta(\overline{x}, y)\rangle \leqslant 0, \quad \text{for all } y \in C. \tag{1}$$

The vector $\overline{x}$ is called a *strong solution* of Problem 1. We denote by $S(P1)$ the set of all such vectors $\overline{x}$.

PROBLEM 2. Find $\overline{x} \in C$ such that for each $y \in C$, there exists $\overline{s} \in T(\overline{x})$ such that

$$\langle \theta(\overline{x}, \overline{s}), \eta(\overline{x}, y)\rangle \leqslant 0. \tag{2}$$

The solution $\overline{x}$ of this problem is called a *weak solution* of Problem 1. We denote by $S(P2)$ the set of all solutions of this problem.

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PROBLEM 3. Find $\overline{x} \in C$ such that

$$\langle \theta(y, t), \eta(\overline{x}, y) \rangle \leqslant 0, \quad \text{for all} \quad y \in C \text{ and } t \in T(y). \tag{3}$$

We denote by $S(P3)$ the set of all its solutions.

Inequalities (1), (2) and (3) are known as *generalised variational-like inequalities* (in short, GVLI). Problem 1 was introduced by Parida and Sen [**13**] in finite dimensional spaces. They also showed its relation with convex mathematical programming. It was further studied by Yao [**19, 20**] with applications in complementarity problems.

When $\theta(x, s) = s$, for any $x \in C$, Problem 1 was considered by Boss [**1**], Ding [**6**] and Siddiqi et al [**17**].

When $\theta(x, s) = s$ and $\eta(x, y) = x - y$, for any $x, y \in C$ and $s \in T(x)$, the above three problems were studied by Crouzeix [**5**] in the setting of finite dimensional spaces. In this case, Problem 1 was studied for example by Browder [**2**], Chowdhury and Tan [**3, 4**], Ding and Tarafdar [**7**], Fang and Peterson [**9**], Saigal [**14**], Shih and Tan [**15**], Siddiqi and Ansari [**16**], Tan [**18**], Yao [**21**], and Yen [**22**].

In Section 2, we first prove that $S(P1) = S(P2) = S(P3)$ under certain conditions. Then we define a gap function [**10**], which provides an optimisation problem formulation, for the generalised variational-like inequality (GVLI)(3). In Section 3, we consider a more general problem which includes Problem 2 as a special case.

Let C and K be nonempty subsets of E and E^*, respectively. Let $\varphi : K \times C \times C \to \mathbb{R}$ be a function and $T : C \to 2^K$ be a multifunction. Then we consider the following problem known as a *generalised implicit variational problem:*

(GIVP) Find $\overline{x} \in C$ such that for each $y \in C$, there exists $\overline{s} \in T(\overline{x})$ such that

$$\varphi(\overline{s}, \overline{x}, y) \leqslant 0. \tag{4}$$

We prove the existence of its solution by using a result of Chowdhury and Tan [**3**] which is a generalised form of the Fan-KKM Theorem [**8**]. As an application, we use our results to prove the existence of solutions of (GVLI).

Let X, Y be subsets of a vector space E such that $\text{co(X)} \subset Y$. Then the multifunction $F : X \to 2^Y$ is called a KKM-*map* if for each $A \in \mathcal{F}(X)$, $\text{co(A)} \subset \bigcup_{x \in A} F(x)$.

The *graph of* F, denoted by $\mathcal{G}(F)$, is

$$\mathcal{G}(F) = \{(x, y) \in X \times Y : x \in X, \ y \in F(x)\}.$$

We shall use the following result of Chowdhury and Tan [**3**] in proving our main results in Section 3.

Theorem A. *Let C be a nonempty convex set in a topological vector space E. Let $G : C \to 2^C$ be a KKM-map such that*

(i) *$\overline{G(y_0)}$ is compact for some $y_0 \in C$,*

(ii) *for each $A \in \mathcal{F}(C)$ with $y_0 \in A$ and each $y \in \text{co(A)}$, $G(y) \cap \text{co(A)}$ is closed in co(A), and*

(iii) *for each $A \in \mathcal{F}(C)$ with $y_0 \in A$,*

$$\overline{\Big(\bigcap_{y \in \text{co(A)}} G(y)\Big)} \cap \text{co(A)} = \Big(\bigcap_{y \in \text{co(A)}} G(y)\Big) \cap \text{co(A)}.$$

Then $\bigcap_{y \in C} G(y) \neq \emptyset$.

The following Kneser minimax theorem [12] will be used in Section 2.

Theorem B. *Let X be a nonempty convex subset of a vector space, and let Y be a nonempty compact convex subset of a Hausdorff topological vector space. Suppose that the functional $f : X \times Y \to \mathbb{R}$ is such that, for each fixed $x \in X$, $f(x, \cdot)$ is lower semicontinuous and convex, and for each fixed $y \in Y$, $f(\cdot, y)$ is concave. Then*

$$\min_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \min_{y \in Y} f(x, y).$$

2. A gap function for (GVLI)

Throughout in this paper, unless specified otherwise, E is a topological vector space with dual E^*.

Let C be a nonempty convex subset of E and K be a nonempty subset of E^*. Given two functions $\theta : C \times K \to E^*$ and $\eta : C \times C \to E$, the multifunction $T : C \to 2^K$ is called:

(i) *η-pseudomonotone with respect to θ* if for every pair of points $x \in K$, $y \in K$ and for all $s \in T(x)$, $t \in T(y)$, we have

$$\langle \theta(x, s), \eta(x, y) \rangle \leqslant 0 \text{ implies } \langle \theta(y, t), \eta(x, y) \rangle \leqslant 0;$$

(ii) *V-hemicontinuous with respect to θ and η* if for all $x, y \in K$, $0 < \lambda < 1$ and $s_\lambda \in T\big(\lambda y + (1 - \lambda)x\big)$, there exists $s \in T(x)$ such that $\langle \theta(x, s_\lambda), \eta(x, y) \rangle$ converges to $\langle \theta(x, s), \eta(x, y) \rangle$ as λ tends to 0^+.

It is clear that $S(P1) \subseteq S(P2)$. By using Theorem B, we prove $S(P2) \subseteq S(P1)$.

Proposition 1. *Let E be a Hausdorff topological vector space with dual E^* and let C and K be nonempty convex subsets of E and E^*, respectively. Let $T : C \to 2^K$ be a compact convex valued multifunction. Assume that*

(a) *for each $x, y \in C$, $s \mapsto \langle \theta(x, s), \eta(x, y) \rangle$ is lower semicontinuous and convex;*

(b) *for each $x \in K$ and $s \in T(x)$, $y \mapsto \langle \theta(x, s), \eta(x, y) \rangle$ is concave.*

Then $S(P2) \subseteq S(P1)$.

PROOF: Let $\bar{x} \in C$ be a solution of Problem 2. Then for each $y \in C$, there exists $\bar{s} \in T(\bar{x})$ such that

$$\langle \theta(\bar{x}, \bar{s}),\ \eta(\bar{x}, y) \rangle \leqslant 0.$$

Define a functional $f : C \times T(\bar{x}) \to \mathbb{R}$ by

$$f(y, s) = \langle \theta(\bar{x}, s),\ \eta(\bar{x}, y) \rangle.$$

By assumption (a), for each $y \in C$, the functional $s \mapsto f(y, s)$ is lower semicontinuous and convex, and by assumption (b), for each $s \in T(\bar{x})$, the functional $y \mapsto f(y, s)$ is concave. Then by Theorem B, we have

$$\begin{aligned}\min_{s \in T(\bar{x})} \sup_{y \in C} \langle \theta(\bar{x}, s),\ \eta(\bar{x}, y) \rangle &= \sup_{y \in C} \min_{s \in T(\bar{x})} \langle \theta(\bar{x}, s),\ \eta(\bar{x}, y) \rangle \\ &= \sup_{y \in C} \Big[\inf_{s \in T(\bar{x})} \langle \theta(\bar{x}, s),\ \eta(\bar{x}, y) \rangle \Big] \\ &\leqslant 0.\end{aligned}$$

Since $T(\bar{x})$ is compact, there exists a point $\bar{s} \in T(\bar{x})$ such that

$$\sup_{y \in C} \Big[\langle \theta(\bar{x}, \bar{s}),\ \eta(\bar{x}, y) \rangle \Big] \leqslant 0,$$

and hence

$$\langle \theta(\bar{x}, \bar{s}),\ \eta(\bar{x}, y) \rangle \leqslant 0, \quad \text{for all} \quad y \in C,$$

that is, $\bar{x} \in S(P1)$. ∎

PROPOSITION 2. *Let C and K be nonempty subsets of E and E^*, respectively. If $T : C \to 2^K$ is η-pseudomonotone with respect to θ, then $S(P1) \subseteq S(P3)$.*

PROPOSITION 3. *Let C be a nonempty convex subset of E and K be a nonempty subset of E^*. Let $\theta(\cdot, \cdot)$ and $\eta(\cdot, \cdot)$ be concave in their first and second arguments, respectively, such that $\eta(x, x) = 0$ for all $x \in C$. If $T : C \to 2^K$ is V-hemicontinuous with respect to θ and η, then $S(P3) \subseteq S(P2)$.*

PROOF: Let $\bar{x} \in S(P3)$. Then

$$\langle \theta(y, t),\ \eta(\bar{x}, y) \rangle \leqslant 0, \quad \text{for all} \quad y \in C \quad \text{and} \quad t \in T(y).$$

By the convexity of C, for any $\lambda \in (0, 1)$, we have

$$\big\langle \theta\big(\lambda y + (1 - \lambda)\bar{x}, s_\lambda\big),\ \eta\big(\bar{x}, \lambda y + (1 - \lambda)\bar{x}\big) \big\rangle \leqslant 0, \quad \text{for all} \quad s_\lambda \in T\big(\lambda y + (1 - \lambda)\bar{x}\big).$$

Since $\theta(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$ are concave in their first and second arguments, respectively, and $\eta(x,x)=0$ for all $x\in C$, we have

$$\begin{aligned}0 &\geqslant \langle \theta(\lambda y+(1-\lambda)\overline{x}, s_\lambda),\ \eta(\overline{x}, \lambda y+(1-\lambda)\overline{x})\rangle \\ &\geqslant \lambda^2\langle \theta(y,s_\lambda),\ \eta(\overline{x},y)\rangle + (1-\lambda)\lambda\langle \theta(\overline{x},s_\lambda),\ \eta(\overline{x},y)\rangle.\end{aligned}$$

Dividing by $\lambda>0$, we get

$$0\geqslant \lambda\langle \theta(y,s_\lambda),\ \eta(\overline{x},y)\rangle + (1-\lambda)\langle \theta(\overline{x},s_\lambda),\ \eta(\overline{x},y)\rangle.$$

Taking $\lambda\to 0^+$ and by V-hemicontinuity with respect to θ and η of T, there exists $\overline{s}\in T(\overline{x})$ such that

$$\langle \theta(\overline{x},\overline{s}),\ \eta(\overline{x},y)\rangle \leqslant 0,$$

and hence $\overline{x}\in S(P2)$. □

By combining Propositions 1-3, we have the following result.

THEOREM 1. *Let E be a Hausdorff topological vector space with dual E^* and let C and K be nonempty convex subsets of E and E^*, respectively. Let $T: C\to 2^K$ be compact convex valued, η-pseudomonotone with respect to θ and V-hemicontinuous with respect to θ and η. Let $\theta(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$ be concave in their first and second arguments, respectively, such that $\eta(x,x)=0$ for all $x\in C$. Let $s\mapsto\langle\theta(x,s),\ \eta(x,y)\rangle$, for all $x,y\in C$, be lower semicontinuous and convex. Then $S(P1)=S(P2)=S(P3)$.*

Let C be a nonempty subset of E. Then a functional $f: C\to\mathbb{R}\cup\{-\infty,+\infty\}$ is called *a gap function* for (GVLI) if

(i) $f(x)\geqslant 0$, for all $x\in C$,

(ii) $f(x)=0$ if and only if x is a solution of (GVLI).

Now, we define a functional $g: C\to\mathbb{R}\cup\{-\infty,+\infty\}$ as follows:

$$g(x)=\sup\Big[\langle\theta(y,t),\ \eta(x,y)\rangle : y\in C \ \text{ and } \ t\in T(y)\Big]. \tag{5}$$

We also set

$$m=\inf_{x\in C} g(x) \quad\text{and}\quad M=\{x\in C: g(x)=m\}.$$

THEOREM 2. *Let C be a nonempty subset of E and let $\eta(x,x)=0$ for all $x\in C$. Then g as defined by (5) is a gap function for (GVLI)(3).*

PROOF: (i) Since $\langle\theta(x,s),\ \eta(x,x)\rangle=0$ for all $x\in C$ and $s\in T(x)$, we have

$$g(x)\geqslant 0, \quad \text{for all} \quad x\in C. \tag{6}$$

(ii) Suppose that $\overline{x}\in C$ is a solution of (GVLI)(3), then

$$\langle\theta(y,t),\ \eta(\overline{x},y)\rangle\leqslant 0, \quad\text{for all}\quad t\in T(y),$$

and hence

$$\sup\left[\langle \theta(y,t),\ \eta(\overline{x},y)\rangle : y \in C \quad \text{and} \quad t \in T(y)\right] \leqslant 0. \tag{7}$$

This implies that $g(\overline{x}) \leqslant 0$. Combining (6) and (7) we get

$$g(\overline{x}) = 0. \tag{8}$$

Conversely, let $g(\overline{x}) = 0$. From (5), we have

$$g(\overline{x}) \geqslant \langle \theta(y,t),\ \eta(\overline{x},y)\rangle, \quad \text{for all} \quad y \in C \quad \text{and} \quad t \in T(y)$$

and hence

$$\langle \theta(y,t),\ \eta(\overline{x},y)\rangle \leqslant 0, \quad \text{for all} \quad y \in C \quad \text{and} \quad t \in T(y).$$

Therefore, $\overline{x} \in C$ is a solution of (GVLI)(3). ∎

THEOREM 3. *Let C be nonempty subset of E and let $\eta(x,x) = 0$, for all $x \in C$. If $S(P3) \neq \emptyset$, then $m = 0$ and $M = S(P3)$.*

PROOF: Let $S(P3) \neq \emptyset$. Then from (8), $m = 0$.

Let $\overline{x} \in C$ be a solution of (GVLI)(3). Then $g(\overline{x}) = 0$. But from (6), we have $g(x) \geqslant 0$ for all $x \in C$, and hence $g(\overline{x}) \leqslant g(x)$ for all $x \in C$. Therefore, $\overline{x} \in M$.

Conversely, assume that $\overline{x} \in M$. Then $g(\overline{x}) = 0$ and thus $\overline{x} \in S(P3)$. Hence $M = S(P3)$. ∎

Combining Theorems 1–3, we have the following result.

THEOREM 4. *Assume that all the hypotheses of Theorem 1 are satisfied and if $m = 0$ and $M \neq \emptyset$, then $M = S(P1) = S(P2) = S(P3)$.*

3. EXISTENCE RESULTS

We first prove the existence of solution of (GIVP) by using Theorem A.

THEOREM 5. *Let C be a nonempty convex subset of E and K be a nonempty subset of E^*. Let $\varphi : K \times C \times C \to \mathbb{R}$ be a function and $T : C \to 2^K$ be a multifunction. Assume that*

1^0 *for each $A \in \mathcal{F}(C)$ and each $x \in \mathrm{co}(A)$, $\min_{y \in A} \varphi(s,x,y) \leqslant 0$ for all $s \in T(x)$;*

2^0 *for each $A \in \mathcal{F}(C)$ and each $y \in \mathrm{co}(A)$,*

$$G(y) \cap \mathrm{co}(A) = \{x \in \mathrm{co}(A) : \text{ there exists } s \in T(x) \text{ such that } \varphi(s,x,y) \leqslant 0\}$$

is closed in $\mathrm{co}(A)$;

3^0 *for each $A \in \mathcal{F}(C)$ and each $x^*, y \in \mathrm{co}(A)$ and for every net $\{x_\alpha\}_{\alpha\in\Gamma}$ in C converging to x^*, if there exists a net $\{s_\alpha\}$ in K with $s_\alpha \in T(x_\alpha)$ for all $\alpha \in \Gamma$, for which*

$$\varphi(s_\alpha, x_\alpha, y) \leqslant 0, \quad \textit{for all} \quad \alpha \in \Gamma,$$

then there exists $s^* \in T(x^*)$ such that $\varphi(s^*, x^*, y) \leqslant 0$;

4^0 *there exists a nonempty closed and compact subset D of C and $z \in D$ such that*

$$\varphi(s', x', z) > 0, \quad \textit{for all} \quad x' \in C\backslash D \textit{ and } s' \in T(x').$$

Then there exists $\overline{x} \in D$ such that for each $y \in C$, there exists $\overline{s} \in T(\overline{x})$ such that $\varphi(\overline{s}, \overline{x}, y) \leqslant 0$.

PROOF: We define the multifunction $G : C \to 2^C$ by

$$G(y) = \{x \in C : \text{ there exists } s \in T(x) \text{ such that } \varphi(s, x, y) \leqslant 0\}, \quad \text{for each} \quad y \in C.$$

We show first that G is a KKM-map.

Suppose that G is not a KKM-map. Then for some finite subset $\{y_1, \dots, y_n\}$ of C and $\lambda_i \geqslant 0$ for all $i = 1, \dots, n$ with $\sum_{i=1}^{n} \lambda_i = 1$, we have $x_0 = \sum_{i=1}^{n} \lambda_i y_i \notin \bigcup_{i=1}^{n} G(y_i)$. Then, for all $s_0 \in T(x_0)$,

$$\varphi(s_0, x_0, y_i) > 0, \quad \text{for all} \quad i = 1, \dots, n$$

and so

$$\min_{1\leqslant i\leqslant n} \varphi(s_0, x_0, y_i) > 0,$$

which contradicts the assumption 1^0. Hence G is a KKM-map. Moreover, we have,

(i) $G(z) \subset D$ by assumption 4^0, so that $\overline{G(z)} \subset \overline{D} = D$ and hence $\overline{G(z)}$ is compact in C;

(ii) for each $A \in \mathcal{F}(C)$ with $z \in A$ and each $y \in \mathrm{co}(A)$,

$$G(y) \cap \mathrm{co}(A) = \{x \in \mathrm{co}(A) : \text{ there exists } s \in T(x) \text{ such that } \varphi(s, x, y) \leqslant 0\}$$

is closed in co(A) by assumption 2^0.

(iii) for each $A \in \mathcal{F}(C)$ with $z \in A$, if $x^* \in \overline{\left(\bigcap_{y\in \mathrm{co}(A)} G(y)\right)} \cap \mathrm{co}(A)$, then $x^* \in \overline{\left(\bigcap_{y\in \mathrm{co}(A)} G(y)\right)}$ and $x^* \in \mathrm{co}(A)$, and there is a net $\{x_\alpha\}$ in $\bigcap_{y\in \mathrm{co}(A)} G(y)$ such that x_α converges to x^*. For each $y \in \mathrm{co}(A)$, there exists a net $\{s_\alpha\}$ in K with $s_\alpha \in T(x_\alpha)$ for which

$$\varphi(s_\alpha, x_\alpha, y) \leqslant 0, \text{ for all } \alpha \in \Gamma.$$

From assumption 3^0, there exists $s^* \in T(x^*)$ such that $\varphi(s^*, x^*, y) \leqslant 0$. It follows that $x^* \in \left(\bigcap_{y \in \mathrm{co}(A)} G(y)\right) \cap \mathrm{co}(A)$ and hence

$$\overline{\left(\bigcap_{y \in \mathrm{co}(A)} G(y)\right)} \cap \mathrm{co}(A) = \left(\bigcap_{y \in \mathrm{co}(A)} G(y)\right) \cap \mathrm{co}(A).$$

By Theorem A, we have $\bigcap_{y \in C} G(y) \neq \emptyset$. Therefore, noting that $\bigcap_{y \in C} G(y) \subseteq G(z) \subseteq D$, there exists $\overline{x} \in D$ such that for each $y \in C$, there exists $\overline{s} \in T(\overline{x})$ such that $\varphi(\overline{s}, \overline{x}, y) \leqslant 0$. ▯

THEOREM 6. *Let C be a nonempty convex subset of E and K be a nonempty compact subset of E^*. Let $\varphi : K \times C \times C \to \mathbb{R}$ be a function and $T : C \to 2^K$ be a multifunction such that its graph is closed. Assume that*

1^0 *for each $A \in \mathcal{F}(C)$ and each $x \in \mathrm{co}(A)$, $\min_{y \in A} \varphi(s, x, y) \leqslant 0$ for all $s \in T(x)$;*

2^0 *for each $A \in \mathcal{F}(C)$ and each $y \in \mathrm{co}(A)$, $\varphi(\cdot, \cdot, y)$ is lower semicontinuous on $K \times \mathrm{co}(A)$;*

3^0 *for each $A \in \mathcal{F}(C)$ and each $x^*, y \in \mathrm{co}(A)$ and for every net $\{x_\alpha\}_{\alpha \in \Gamma}$ in C converging to x^*, if there exists a net $\{s_\alpha\}$ in K with $s_\alpha \in T(x_\alpha)$ for all $\alpha \in \Gamma$, for which*

$$\varphi(s_\alpha, x_\alpha, y) \leqslant 0 \text{ for all } \alpha \in \Gamma,$$

then there exists $x^* \in T(x^*)$ such that $\varphi(s^*, x^*, y) \leqslant 0$;

4^0 *there exists a nonempty closed and compact subset D of C and $z \in D$ such that*

$$\varphi(s', x', z) > 0, \quad \text{for all } \ y \in C \backslash D \text{ and } s' \in T(x').$$

Then there exists $\overline{x} \in D$ such that for each $y \in C$, there exists $\overline{s} \in T(\overline{x})$ such that $\varphi(\overline{s}, \overline{x}, y) \leqslant 0$.

PROOF: If we prove that for each $A \in \mathcal{F}(C)$ with $z \in A$ and each $y \in \mathrm{co}(A)$,

$$G(y) \cap \mathrm{co}(A) = \{x \in \mathrm{co}(A) : \text{ there exists } s \in T(x) \text{ such that } \varphi(s, x, y) \leqslant 0\}$$

is closed in co(A) then from Theorem 5, we get the result.

Indeed, let $\{x_\beta\}_{\beta \in \Lambda}$ be a net in $G(y) \cap \mathrm{co}(A)$ such that x_β converges to x. Then $x \in \mathrm{co}(A)$, because co(A) is compact (see [3, p.922]). Since $x_\beta \in G(y) \cap \mathrm{co}(A)$, there exist $s_\beta \in T(x_\beta)$ such that $\varphi(s_\beta, x_\beta, y) \leqslant 0$. Since $T(C)$ is contained in a compact set

K, we may assume that s_β converges to some $s \in K$. Then from the closed graph of T, we have $s \in T(x)$. Since $\varphi(\cdot, \cdot, y)$, for each $y \in \mathrm{co}(A)$, is lower semicontinuous, we get

$$0 \geqslant \liminf_\beta \varphi(s_\beta, x_\beta, y) \geqslant \varphi(s, x, y)$$

and hence $x \in G(y) \cap \mathrm{co}(A)$, as desired. □

As applications of Theorem 5 and Theorem 6, we have the following results:

Corollary 1. *Let C be a nonempty convex subset of E and K be a nonempty subset of E^*. Let $\theta : C \times K \to E^*$ and $\eta : C \times C \to E$ be functions and $T : C \to 2^K$ be a multifunction. Assume that*

1^0 *for each $A \in \mathcal{F}(C)$ and each $x \in \mathrm{co}(A)$, $\min_{y \in A} \langle \theta(x, s), \eta(x, y) \rangle \leqslant 0$ for all $s \in T(x)$;*

2^0 *for each $A \in \mathcal{F}(C)$ and each $y \in \mathrm{co}(A)$, the set*

$$\{x \in \mathrm{co}(A) : \text{there exists } s \in T(x) \text{ such that } \langle \theta(x, s), \eta(x, y) \rangle \leqslant 0\}$$

is closed in $\mathrm{co}(A)$;

3^0 *for each $A \in \mathcal{F}(C)$ and each $x^*, y \in \mathrm{co}(A)$ and for every net $\{x_\alpha\}_{\alpha \in \Gamma}$ in C converging to x^*, if there exists a net $\{s_\alpha\}$ in K with $s_\alpha \in T(x_\alpha)$ for all $\alpha \in \Gamma$, for which*

$$\langle \theta(x_\alpha, s_\alpha), \eta(x_\alpha, y) \rangle \leqslant 0, \quad \text{for all } \alpha \in \Gamma,$$

then there exists $s^* \in T(x^*)$ such that $\langle \theta(x^*, s^*), \eta(x^*, y) \rangle \leqslant 0$;

4^0 *there exists a nonempty closed and compact subset D of C and $z \in D$ such that*

$$\langle \theta(x', s'), \eta(x', z) \rangle > 0, \quad \text{for all } y \in C \backslash D \text{ and } s' \in T(x').$$

Then there exists $\overline{x} \in D$ such that for each $y \in C$, there exists $\overline{s} \in T(\overline{x})$ such that

$$\langle \theta(\overline{x}, \overline{s}), \eta(\overline{x}, y) \rangle \leqslant 0.$$

Proof: By taking $\varphi(s, x, y) = \langle \theta(x, s), \eta(x, y) \rangle$ in Theorem 5, we get the result. □

Corollary 2. *Let C be a nonempty convex subset of E and K be a nonempty compact subset of E^*. Let $\theta : C \times K \to E^*$ and $\eta : C \times C \to E$ be functions and $T : C \to 2^K$ be a multifunction such that its graph is closed. Assume that*

1^0 *for each $A \in \mathcal{F}(C)$ and each $x \in \mathrm{co}(A)$, $\min_{y \in A} \langle \theta(x, s), \eta(x, y) \rangle \leqslant 0$ for all $s \in T(x)$;*

2⁰ *for each* $A \in \mathcal{F}(C)$ *and each* $y \in \text{co}(A)$, $\langle\theta(x,s),\ \eta(x,y)\rangle$ *is lower semicontinuous in* $(s,x) \in K \times \text{co}(A)$;

3⁰ *for each* $A \in \mathcal{F}(C)$ *and each* $x^*, y \in \text{co}(A)$ *and for every net* $\{x_\alpha\}_{\alpha\in\Gamma}$ *in* C *converging to* x^*, *if there exists a net* $\{s_\alpha\}$ *in* K *with* $s_\alpha \in T(x_\alpha)$ *for all* $\alpha \in \Gamma$, *for which*

$$\langle\theta(x_\alpha, s_\alpha),\ \eta(x_\alpha, y)\rangle \leqslant 0, \quad \text{for all} \quad \alpha \in \Gamma,$$

then there exists $s^* \in T(x^*)$ *such that* $\langle\theta(x^*, s^*),\ \eta(x^*, y)\rangle \leqslant 0$;

4⁰ *there exists a nonempty closed and compact subset* D *of* C *and* $z \in D$ *such that*

$$\langle\theta(x', s'),\ \eta(x', z)\rangle > 0, \quad \text{for all} \quad y \in C\backslash D \text{ and } s' \in T(x').$$

Then there exists $\overline{x} \in D$ *such that for each* $y \in C$, *there exists* $\overline{s} \in T(\overline{x})$ *such that*

$$\langle\theta(\overline{x}, \overline{s}),\ \eta(\overline{x}, y)\rangle \leqslant 0.$$

PROOF: By taking $\varphi(s,x,y) = \langle\theta(x,s),\ \eta(x,y)\rangle$ in Theorem 6, we get the result. ∎

COROLLARY 3. *Let* C *be a nonempty convex subset of* E *and* K *be a nonempty compact subset of* E^*. *Let* $\theta : C \times K \to E^*$ *and* $\eta : C \times C \to E$ *be functions and* $T : C \to 2^K$ *be a multifunction such that its graph is closed. Assume that*

1⁰ $\langle\theta(x,s),\ \eta(x,x)\rangle = 0$ *for all* $x \in C$ *and* $s \in T(x)$;

2⁰ $y \mapsto \langle\theta(x,s),\ \eta(x,y)\rangle$ *is quasiconcave for each fixed* $x \in C$ *and* $s \in T(x)$;

3⁰ *for each* $A \in \mathcal{F}(C)$ *and each* $y \in \text{co}(A)$, $\langle\theta(x,s),\ \eta(x,y)\rangle$ *is lower semicontinuous in* $(s,x) \in K \times \text{co}(A)$;

4⁰ *for each* $A \in \mathcal{F}(C)$ *and each* $x^*, y \in \text{co}(A)$ *and for every net* $\{x_\alpha\}_{\alpha\in\Gamma}$ *in* C *converging to* x^*, *if there exists a net* $\{s_\alpha\}$ *in* K *with* $s_\alpha \in T(x_\alpha)$ *for all* $\alpha \in \Gamma$, *for which*

$$\langle\theta(x_\alpha, s_\alpha),\ \eta(x_\alpha, y)\rangle \leqslant 0, \quad \text{for all} \quad \alpha \in \Gamma,$$

then there exists $s^* \in T(x^*)$ *such that* $\langle\theta(x^*, s^*),\ \eta(x^*, y)\rangle \leqslant 0$;

5⁰ *there exists a nonempty closed and compact subset* D *of* C *and* $z \in D$ *such that*

$$\langle\theta(x', s'),\ \eta(x', z)\rangle > 0, \quad \text{for all} \quad y \in C\backslash D \text{ and } s' \in T(x').$$

Then there exists $\overline{x} \in D$ *such that for each* $y \in C$, *there exists* $\overline{s} \in T(\overline{x})$ *such that*

$$\langle\theta(\overline{x}, \overline{s}),\ \eta(\overline{x}, y)\rangle \leqslant 0.$$

PROOF: In view of assumptions 1⁰ and 2⁰, it is easy to prove that the multifunction G in the proof of Theorem 5 is a KKM-map. By taking $\theta(s,x,y) = \langle\theta(x,s),\ \eta(x,y)\rangle$ in Corollary 2, we get the result. ∎

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