Ann. Sci. Math. Québec **21** (1997), no. 1, 1–11.

EXTENDED GENERALIZED VECTOR VARIATIONAL-LIKE INEQUALITIES FOR NONMONOTONE MULTIVALUED MAPS

QAMRUL HASAN ANSARI

RÉSUMÉ. Dans le présent travail, nous étudions une extension du problème vectoriel généralisé d'inéquations de type variationnel et nous démontrons l'existence de sa solution dans le contexte des espaces vectoriels topologiques. Divers cas spéciaux sont aussi traités.

ABSTRACT. In this paper, we study an extension of generalized vector variational-like inequality and prove the existence of its solution in the setting of Hausdorff topological vector spaces. Several special cases were also discussed.

1. Introduction. The vector variational inequality was introduced by Giannessi [8] in the finite dimensional Euclidean space in 1980 and has a lot of applications in vector optimization, Since then it has been generalized in various directions. Ansari [1], Chen *et al.* [2–6], Siddiqi *et al.* [12, 13], Lee *et al.* [11] and Yang [14] have studied vector variational inequalities in abstract spaces. Variational inequalities and variational-like inequalities for multivalued maps have been shown to be a useful tool in different areas of optimization, optimal control, operations research and economics. Huang [9] and Jou and Yao [10] studied a more general form of generalized variational inequalities. Recently, vector variational-like inequality for multivalued maps has been studied in [1]. Inspired and motivated by the applications of vector variational and vector variational-like inequalities for multivalued maps, in this paper, we introduce an extension of generalized variational inequality [9, 10] and generalized vector variational-like inequalities [1] and prove the existence of its solution in the setting of Hausdorff topological vector spaces. Several special cases were also discussed.

Let X and Y be two Hausdorff topological vector spaces and K be a nonempty convex subset of X. Let $T : K \to 2^{L(X,Y)}$ be a multivalued map, where $L(X, Y)$ is the space of all linear continuous maps from X into Y. Let $\eta : K \times K \to X$ and $A : L(X,Y) \to L(X,Y)$ be continuous maps and $\{C(x) : x \in K\}$ be a family of closed pointed convex cones in Y with $int\, C(x) \neq \emptyset$ for every $x \in K$, where $int\, C(x)$ is the interior of the set $C(x)$.

We consider the problem of finding $x_0 \in K$ such that for each $x \in K$, there exists $w_0 \in T(x_0)$ such that

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0),$$

Reçu le 7 février 1996 et, sous forme définitive, le 25 juin 1996.

where $\langle Aw_0, y\rangle$ denotes the evaluation of the linear map Aw_0 at y. We shall call it *extended generalized vector variational-like inequality problem* (EGVVLIP).

Special cases.

(i) If $Y = \mathbb{R}$, $L(X, Y) = X^*$, the dual space of X, $C(x) = \mathbb{R}_+$ for all $x \in K$ and $\eta(x, x_0) = x - x_0$ then the EGVVLIP reduces to the problem of finding $x_0 \in K$ such that for each $x \in K$, there exists $w_0 \in T(x_0)$ such that

$$\langle Aw_0, x - x_0\rangle \geq 0,$$

which is called *extended generalized variational inequality problem*, studied by Huang [9] and Jou and Yao [10].

(ii) If A is an identity map then the EGVVLIP becomes to the problem of finding $x_0 \in K$ such that for each $x \in K$, there exists $w_0 \in T(x_0)$ such that

$$\langle w_0, \eta(x, x_0)\rangle \notin -int\, C(x_0),$$

which is known as *generalized vector variational-like inequality problem* [1].

(iii) If A is an identity map and $\eta(x, x_0) = x - x_0$ then EGVVLIP is equivalent to the following *generalized vector variational inequality problem* considered by Lee *et al.* [11] and Chen and Craven [5]:

Find $x_0 \in K$ such that for each $x \in K$, there exists $w_0 \in T(x_0)$ such that

$$\langle w_0, x - x_0\rangle \notin -int\, C(x_0).$$

(iv) If A is an identity map and T is a single-valued map then the EGVVLIP reduces to the problem of finding $x_0 \in K$ such that for each $x \in K$,

$$\langle T(x_0), \eta(x, x_0)\rangle \notin -int\, C(x_0),$$

which is called *vector variational-like inequality problem* [12].

(v) If A is an identity map, T is a single-valued map and $\eta(x, x_0) = x - x_0$ then the EGVVLIP is equivalent to the problem of finding $x_0 \in K$ such that for each $x \in K$,

$$\langle T(x_0), x - x_0\rangle \notin -int\, C(x_0),$$

which is known as *vector variational inequality problem* [2].

2. Existence theory. Through out in this paper, we will consider X and Y as Hausdorff topological vector spaces. We denote 2^X the set of all nonempty subsets of X, $conv(A)$, for all $A \subset X$, the convex hull of A and $\partial(A)$, the boundary of A. For $K, B \subseteq X$, $int_K(B)$ and $\partial_K(B)$ denote the relative interior and relative boundary of B in K, respectively. The bilinear form $\langle \cdot, \cdot\rangle$ is supposed to be continuous.

We need the following concepts and results to prove our main results of this paper.

Definition 2.1. [9] A multivalued map $T : X \to 2^Y$ is said to be *uniformly compact near* $x \in X$ (or locally bounded at $x \in X$) if there exists a neighborhood U of x in X such that $T(U) = \bigcup_{y \in U} T(y)$ is bounded.

If T is uniformly compact near x for any $x \in X$, then T is said to be *uniformly compact* on X.

Definition 2.2. A multivalued map $T : X \to 2^X$ is called *KKM-map*, if for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, $conv(\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} T(x_i)$.

Lemma 2.1. (KKM – Fan [7]) *Let A be an arbitrary nonempty set in a topological vector space E and $T : A \to 2^E$ be a KKM-map. If $T(x)$ is closed for all $x \in A$ and is compact for at least one $x \in A$ then*

$$\bigcap_{x \in A} T(x) \neq \emptyset.$$

Lemma 2.2 [7]. *Let K be a nonempty compact convex set in a Hausdorff topological vector space X and A be a subset of $K \times K$ having the following properties:*

1" *for every $x \in K$, $(x, x) \in A$,*
2" *for each fixed $x \in K$, the set $A_x = \{y \in K : (x, y) \in A\}$ is closed in K,*
3" *for each fixed $y \in K$, the set $A_y = \{x \in K : (x, y) \notin A\}$ is convex.*

Then there exists a point $x_0 \in K$ such that

$$K \times \{x_0\} \subset A.$$

Lemma 2.3. *Let Y be a topological vector space with a closed, pointed and convex cone P whose interior $int\, P$ is nonempty. Then for all $u, v \in Y$, we have*

$$u + v \notin -int\, P \text{ and } v \in -int\, P \Rightarrow u \notin -int\, P.$$

Proof. Suppose that $u \in -int\, P$. Since $v \in -int\, P$, we have $u+v \in -int\, P - int\, P \subseteq -int\, P$ and hence $u + v \in -int\, P$, which is a contradiction of our assumption. □

Now we prove the existence theorem for the EGVVLIP where T is not a monotone operator.

Theorem 2.1. *Let K be a nonempty compact convex subset of X. Assume that*

1" *$C : K \to 2^Y$ is a multivalued map such that for every $x \in K$, $C(x)$ is a closed pointed convex cone with $int\, C(x) \neq \emptyset$,*
2" *$T : K \to 2^{L(X,Y)}$ is upper semicontinuous and uniformly compact,*
3" *$A : L(X, Y) \to L(X, Y)$ is continuous,*
4° *$\eta : K \times K \to X$ is continuous, and affine in the first argument such that $\eta(x, x) = 0$, for all $x \in K$,*
5" *the multivalued map $W(x) = Y \setminus \{-int C(x)\}$ is upper semicontinuous on K.*

Then there exist $x_0 \in K$ and $w_0 \in T(x_0)$ such that

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0), \text{ for all } x \in K.$$

Proof. Let $A = \{(x, y) \in K \times K : \exists v \in T(y)$ such that $\langle Av, \eta(x, y)\rangle \notin -int\, C(y)\}$.

To prove this theorem, we will show that assumptions 1°, 2" and 3" of Lemma 2.2 are satisfied.

Since $C(x)$ is a closed pointed convex cone and $int\, C(x) \neq \emptyset$, for all $x \in K$, then by assumption 4°, we have

6″ $\langle Aw, \eta(x,x)\rangle \notin -int\, C(x)$, for all $w \in T(x)$ and $x \in K$,

and therefore, from the definition of A, we have $(x, x) \in A$, for all $x \in K$ if and only if

$$\exists w \in T(x) \text{ such that } \langle Aw, \eta(x,x)\rangle \notin -int\, C(x).$$

Now, let $A_x = \{y \in K : (x, y) \in A\}$, for each fixed $x \in K$, then we show that A_x is closed.

Let $\{y_n\}$ be a net in A_x such that $y_n \to y$. Then y is a limit point of A_x and hence $y \in K$ because K is closed. For each y_n in the net, there exists a $v_n \in T(y_n)$ such that for any $z \in K$,

$$\langle Av_n, \eta(x, y_n)\rangle \notin -int\, C(y_n).$$

Hence $\langle Av_n, \eta(x, y_n)\rangle \in W(y_n) = Y \setminus \{-int\, C(y_n)\}$.

Since T is uniformly compact near y, the net $\{v_n\}$ is bounded and thus has a convergent subnet $\{v_{n_k}\}$. We will still denote this subnet by $\{v_n\}$ and its corresponding subnet of $\{y_n\}$ by $\{y_n\}$. Let $w_n \to w$, then by upper semicontinuity of T, we have $v \in T(y)$.

Since A, η and $\langle \cdot, \cdot \rangle$ are continuous, we have

$$\langle Av_n, \eta(x, y_n)\rangle \to \langle Av, \eta(x, y)\rangle.$$

The upper semicontinuity of multivalued map $W(y)$ implies that $\langle Aw, \eta(x, y)\rangle \in W(y)$ and hence $\langle Av, \eta(x, y)\rangle \notin C(y)$, for all $x \in K$. This implies that $y \in A_x$ and hence A_x is closed.

Now we show that for each fixed $y \in K$, the set $A_y = \{x \in K : (x, y) \notin A\}$ is convex.

Indeed, if $x_1, x_2 \in A_y$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$ and since $C(x)$ is a cone, we have

$$\exists v \in T(y) \text{ such that } \langle Av, \eta(x_1, y)\rangle \in -int\, C(y) \tag{1}$$

and

$$\langle Av, \eta(x_2, y)\rangle \in -int\, C(y). \tag{2}$$

Multilying (1) by α and (2) by β and then adding, we get

$$\alpha\langle Av, \eta(x_1, y)\rangle + \beta\langle Av, \eta(x_2, y)\rangle \in -int\, C(y).$$

Since $\eta(\cdot,\cdot)$ is affine in the first argument, we have

$$\langle Av, \eta(\alpha x_1 + \beta x_2, y)\rangle \in -int\, C(y).$$

This implies that $\alpha x_1 + \beta x_2 \in A_y$ and hence A_y is convex.

Then by Lemma 2.2, there exists $x_0 \in K$ such that $K \times \{x_0\} \subset A$, which implies that $x_0 \in K$ and $w_0 \in T(x_0)$ such that

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0), \text{ for all } x \in K. \quad \square$$

When K is not necessarily compact, we have the following existence result.

Theorem 2.2. *Let K be a nonempty convex subset of X. Assume that*

1° $C : K \to 2^Y$ *is a multivalued map such that for every $x \in K$, $C(x)$ is a closed pointed convex cone with $int\, C(x) \neq \emptyset$,*
2° $T : K \to 2^{L(X,Y)}$ *is a multivalued map,*
3° $A : L(X,Y) \to L(X,Y)$ *is continuous,*
4° $\eta : K \times K \to X$ *is a map,*
5° $W : K \to 2^Y$ *is a multivalued map such that $W(x) = Y \setminus \{-int C(x)\}$, for all $x \in K$,*
6° *there exists a nonempty compact convex subset B of K such that*
(i) $T : B \to 2^{L(X,Y)}$ *is upper semicontinuous and uniformly compact,*
(ii) $\eta : B \times B \to X$ *is continuous, and affine in the first argument such that $\eta(x,x) = 0$, for all $x \in B$,*
(iii) *the multivalued map $W : B \to 2^Y$ such that $W(x) = Y \setminus \{-int C(x)\}$, for all $x \in B$, is upper semicontinuous,*
(iv) *for any $x \in \partial_K(B)$, there exists an $x^* \in int_K(B)$ such that for any $w \in T(x)$, $\langle Aw, \eta(x^*, x)\rangle \in -int\, C(x)$.*

Then there exist $x_0 \in K$ and $w_0 \in T(x_0)$ such that

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0), \quad \textit{for all } x \in K.$$

Proof. By Theorem 2.1, there exist $x_0 \in B$ and $w_0 \in T(x_0)$ such that

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0), \quad \text{for all } x \in B. \tag{3}$$

Now, for any $x \in K$, if $x_0 \in int_K(B)$, then there exists an α such that $0 < \alpha < 1$ and $\alpha x + (1-\alpha)x_0 \in B$. Then from (3), we have

$$\langle Aw_0, \eta(\alpha x + (1-\alpha)x_0, x_0)\rangle \notin -int\, C(x_0).$$

Since $\eta(\cdot,\cdot)$ is affine in the first argument and $\eta(x_0, x_0) = 0$, for all $x_0 \in B$ and $C(x_0)$ is a cone, we have

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0). \tag{4}$$

On the other hand, if $x_0 \in \partial_K(B)$ then by assumption 6°(iv), there exists an $x^* \in int_K(B)$ such that for any $w_0 \in T(x_0)$,

$$\langle Aw_0, \eta(x^*, x_0)\rangle \in -int\, C(x_0). \tag{5}$$

Again, choose $0 < \alpha < 1$ such that $\alpha x + (1-\alpha)x^* \in B$. Then from (3), we have

$$\langle Aw_0, \eta(\alpha x + (1-\alpha)x^*, x_0)\rangle \notin -int\, C(x_0).$$

Since $\eta(\cdot,\cdot)$ is affine in the first argument, we have

$$\alpha\langle Aw_0, \eta(x, x_0)\rangle + (1-\alpha)\langle Aw_0, \eta(x^*, x_0)\rangle \notin -int\, C(x_0).$$

Now, since $C(x_0)$ is a cone, then by Lemma 2.3 and (5), we get

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0). \quad \square$$

Coroiiary 2.1. *Let K be a nonempiy convex subset of X. Assume that*

1° $C: K \to 2^Y$ *is a multivalued map such that for every $x \in K$, $C(x)$ is a closed pointed convex cone with $int\, C(x) \neq \emptyset$,*
2" $T: K \to 2^{L(X,Y)}$ *is a multivalued map,*
3" $A: L(X,Y) \to L(X,Y)$ *is continuous,*
4° $\eta: K \times K \to X$ is *a map,*
5" $W: K \to 2^Y$ *is a multivalued map such that $W(x) = Y \setminus \{-intC(x)\}$, for all $x \in K$,*
6" *there exists a subset E of X such that*
(i) $K \cap E$ *is nonempiy compact and convex,*
(ii) $T: K \cap E \to 2^{L(X,Y)}$ *is upper semicontinuous and uniformly compact,*
(iii) $\eta: (K \cap E) \times (K \cap E) \to X$ *is continuous, and affine in the first argument such that $\eta(x,x) = 0$, for all $x \in (K \cap E)$,*
(iv) *the multivalued map $W: (K \cap E) \to 2^Y$ such that $W(x) = Y \setminus \{-int\, C(x)\}$, for all $x \in (K \cap E)$, is upper semicontinuous,*
(v) *for any $x \in \partial_K(K \cap E)$, there exists an $x^* \in int_K(K \cap E)$ such that for any $w \in T(x)$, $\langle Aw, \eta(x^*, x)\rangle \in -int\, C(x)$.*

Then there exist $x_0 \in K$ and $w_0 \in T(x_0)$ such that

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0), \text{ for all } x \in K.$$

Proof. The result follows from Theorem *2.2* by setting $B = K \cap E$. □

Coroiiary 2.2. *Let K be a nonempiy convex subset of X. Assume that* 1°, 2°, 3°, 4°, 5°, 6°*(i),* 6°*(ii),* 6°*(iii),* 6°*(iv) in Corollary 2.1 hold and assume also that*

6°$(v)'$ *for any $x \in K \cap \partial(E)$, there exists an $x^* \in K \cap int(E)$ such that for any $w \in T(x)$, $\langle Aw, \eta(x^*, x)\rangle \in -int\, C(x)$.*

Then there exist $x_0 \in K$ and $w_0 \in T(x_0)$ such that

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0), \text{ for all } x \in K.$$

Proof. Note that $K \cap int(E) \subseteq int_K(K \cap E)$ and $\partial_K(K \cap E) \subseteq K \cap \partial(E)$. Then Condition *6"(u)'* implies Condition *6"(u)* of Corollary *2.1* and we are done. □

Theorem 2.3. *Let K be a nonempiy closed convex subset of X. Assume that*

1° $C: K \to 2^Y$ *is a multivalued map such that for every $x \in K$, $C(x)$ is a closed pointed convex cone with $int\, C(x) \neq 0$,*
2° $T: K \to 2^{L(X,Y)}$ *is upper semicontinuous and uniformly compact,*
3" $A: L(X,Y) \to L(X,Y)$ *is continuous,*
4" $\eta: K \times K \to X$ *is continuous,*
5" *the multivalued map $W(x) = Y \setminus \{-int\, C(x)\}$ is upper semicontinuous on K,*
6" *there exists a function $h: K \times K \to Y$ such that*
(i) *there exists $v \in T(y)$ such that*

$$\langle Av, \eta(x,y)\rangle + h(x,y) \notin -int\, C(y), \text{ for all } (x,y) \in K \times K,$$

(ii) *the set* $\{x \in K : h(x, y) \notin -int\, C(y)\}$ *is convex, for all* $y \in K$,
(iii) $h(x,x) \in -int\, C(x)$, *for all* $x \in K$,
(iv) *there exists a nonempty compact convex subset* D *of* K *such that for every* $y \in K \setminus D$ *there exist* $x \in D$ *and* $v \in T(y)$ *such that*

$$\langle Av, \eta(x,y)\rangle \in -int\, C(y).$$

Then there exist $x_0 \in D \subset K$ *and* $w_0 \in T(x_0)$ *such that*

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0), \quad \text{for all } x \in K.$$

Proof. Let $D(x) = \{y \in D : \exists v \in T(y) \text{ such that } \langle Av, \eta(x,y)\rangle \notin int\, C(y)\}$, for each $x \in K$.

From the proof of Theorem **2.1**, we have $D(x)$ is closed, for each $x \in K$. Since every element $x_0 \in \bigcap_{x\in K} D(x)$ with $w_0 \in T(x_0)$ is a solution of the EGVVLLP, we have to prove that $\bigcap_{x \in K} D(x) \neq 0$.

Since D is compact, it is sufficient to show that the family $\{D(x)\}_{x\in K}$ has finite intersection property.

Let $x_1, x_2, \ldots, x_m \in K$ be given. We put $A = conv\left(D \cup \{x_1, x_2, \ldots, x_m\}\right)$ and we have that A is a compact convex subset of K.

We now consider the following multivalued maps:

$$F_1(x) = \{y \in A : \exists v \in T(y) \text{ such that } \langle Av, \eta(x,y)\rangle \notin -int\, C(y)\}, \text{ for all } x \in K$$

and

$$F_2(x) = \{y \in A : h(x,y) \in -int\, C(y)\}, \text{ for all } x \in K.$$

Again from the proof of Theorem 2.1, $F_1(x)$ is a closed subset of a compact convex set A and hence $F_1(x)$ is compact.

From assumptions 6°(i) and 6°(iii), we have

$$\exists v \in T(y) \text{ such that } \langle Av, \eta(y,y)\rangle + h(y,y) \notin -int\, C(y)$$

and

$$h(y,y) \in -int\, C(y).$$

Then by Lemma **2.3**, we have

$$\langle Av, \eta(y,y)\rangle \notin -int\, C(y)$$

and hence $F_1(x)$ is nonempty.

Now we will prove that F_2 is a KKM-map.

We suppose that there exist $x_1, x_2, \ldots, x_n \in A$ and $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$, with $\sum_{i=1}^n \alpha_i = 1$, such that

$$\sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{i=1}^{n} F_2(x_i)$$

then we have

$$h\left(x_i, \sum_{i=1}^{n} \alpha_i x_i\right) \notin -int\,C\left(\sum_{i=1}^{n} \alpha_i x_i\right).$$

By assumption 6°(ii),

$$h\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \alpha_i x_i\right) \notin -int\,C\left(\sum_{i=1}^{n} \alpha_i x_i\right)$$

which is a contradiction to assumption 6°(iii). Hence F_2 is a KKM-map. From assumption 6°(i), we have $F_2(x) \subset F_1(x)$, for every $x \in K$.

Indeed, let y $\in F_2(x)$ then $h(x, y) \in -int\,C(y)$ and by assumption 6°(i), we have

$$\exists v \in T(y) \text{ such that } \langle Av, \eta(x, y)\rangle + h(x, y) \notin -int\,C(y).$$

By Lemma 2.3, we get

$$\exists v \in T(y) \text{ such that } \langle Av, \eta(x, y)\rangle \notin -int\,C(y).$$

This implies that F_1 is also a KKM-map. Applying Lemma2.1 to F_1, we get $\bigcap_{x\in A} F_1(x)$ # 0, that is, there exists a point $x_0 \in A$ with $w_0 \in T(x_0)$ such that

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\,C(x_0), \text{ for all } x \in \text{A}.$$

By assumption 6°(iv), we have that $x_0 \in D$ with $w_0 \in T(x_0)$ andmoreoverlço $\in D(x_i)$, for every $1 \leq i \leq m$. Hence $\{D(x)\}_{x\in K}$ has the finite intersection property and the proof is finished. □

Résumé substantiel en français. Soient X et Y deux espaces vectoriels topologiques séparés et K un sous-ensemble convexe fermé non vide de X. Soit $T : K \to 2^{L(X,Y)}$ une application multivoque, où $L(X, Y)$ désigne l'espace des fonctions linéaires continues de X dans Y. Soient $\eta : K \times K \to$ X et A $: L(X, Y) \to L(X, Y)$ des applications continues et $\{C(x) : x \in K\}$ une famille de cônes convexes pointés fermés dans Y tels que $int\,C(x) \neq \emptyset$, pour tout $x \in K$, où $int\,C(x)$ désigne l'intérieur de l'ensemble $C(x)$.

Nous considérons le problème de trouver $x_0 \in K$ tel que pour tout $x \in K$, il existe $w_0 \in T(x_0)$ satisfaisant

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\,C(x_0),$$

où $\langle Aw_0, y\rangle$ désigne la valeur de l'application linéaire Aw_0 en y. Ce problème est ici appelé *problème vectoriel généralisé étendu d'inéquations de type variationnel* (en anglais : *extended generalized vector variational-like inequalitiesproblem).*

Nous démontrons, dans un premier temps, un théorème d'existence pour ce problème lorsque T n'est pas un opérateur monotone.

Théorème 2.1. *Soit K un sous-ensemble compact convexe non vide de X. Supposons de plus que*

1) $C : K \to 2^Y$ *est une application multivoque telle que pour tout $x \in K$, $C(x)$ est un cône convexe pointé fermé d'intérieur non vide ;*
2) $T : K \to 2^{L(X,Y)}$ *est semi-continue supérieurement et uniformément compacte ;*
3) $A : L(X,Y) \to L(X,Y)$ *est continue;*
4) $\eta : K \times K \to X$ *est continue, affine selon la première variable et est telle que $\eta(x,x) = 0$ pour tout $x \in K$;*
5) *la fonction multivoque $W(x) = Y \setminus \{-int\, C(x)\}$ est semi-continue supérieurement sur K.*

Alors il existe $x_0 \in K$ et $w_0 \in T(x_0)$ tels que

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0), \text{ pour tout } x \in K.$$

Lorsque K n'est pas nécessairement compact, on a le résultat d'existence suivant.

Théorème 2.2. *Soit K un sous-ensemble convexe non vide de X. Supposons de plus que :*

1) $C : K \to 2^Y$ *est une application multivoque telle que pour tout $x \in K$, $C(x)$ est un cône convexe pointé fermé d'intérieur non vide ;*
2) $T : K \to 2^{L(X,Y)}$ *est multivoque ;*
3) $A : L(X,Y) \to L(X,Y)$ *est continue;*
4) $\eta : K \times K \to X$ *est une application;*
5) *la fonction multivoque $W : K \to 2^Y$ est telle que $W(x) = Y \setminus \{-intC(x)\}$ pour tout $x \in K$;*
6) *il existe un ensemble compact convexe non vide B de K tel que :*

(i) $T : B \to 2^{L(X,Y)}$ *est semi-continue supérieurement et uniformément compacte,*
(ii) $\eta : B \times B \to X$ *est continue, affine selon la première variable et est telle que pour tout $x \in B$, $\eta(x,x) = 0$,*
(iii) *la fonction multivoque $W : B \to 2^Y$, définie par $W(x) = Y \setminus \{-intC(x)\}$, pour tout $x \in B$, est semi-continue supérieurement,*
(iv) *pour tout $x \in \partial_K(B)$, il existe un $x^* \in int_K(B)$ tel que pour tout $w \in T(x)$,*

$$\langle Aw, \eta(x^*, x)\rangle \in -int\, C(x).$$

Alors il existe $x_0 \in K$ et $w_0 \in T(x_0)$ tels que

$$\langle Aw_0, \eta(x, x_0)\rangle \notin -int\, C(x_0), \text{ pour tout } x \in K.$$

Théorème 2.3. *Soit K un sous-ensemble fermé convexe non vide de X. Supposons de plus que :*

1) $C : K \to 2^Y$ *est une application multivoque telle que pour tout $x \in K$, $C(x)$ est un cône convexe pointé fermé d'intérieur non vide ;*

2) $T : K \to 2^{L(X,Y)}$ *est semi-continue supérieurement et uniformément compacte ;*
3) $A : L(X,Y) \to L(X,Y)$ *est continue;*
4) $\eta : K \times K \to X$ *est continue;*
5) la fonction multivoque $W(x) = Y \setminus \{-int\, C(x)\}$ *est semi-continue supérieurement sur* K ;
6) il existe une fonction $h : K \times K \to Y$ *telle que :*
(i) il existe $v \in T(y)$ *tel que*

$$\langle Av, \eta(x,y)\rangle + h(x,y) \notin -int\, C(y), \text{ pour tout } (x,y) \in K \times K,$$

(ii) l'ensemble $\{x \in K : h(x,y) \notin -int\, C(y)\}$ *est convexe, pour tout* $y \in K$,
(iii) $h(x,x) \in -int\, C(x)$, *pour tout* $x \in K$,
(iv) il existe un sous-ensemble non vide compact convexe D *de* K *tel que pour tout* $y \in K \setminus D$ *il existe* $x \in D$ *et* $v \in T(y)$ *tels que*

$$\langle Av, \eta(x,y)\rangle \in -int\, C(y).$$

Alors il existe $x_0 \in D \subset K$ *et* $w_0 \in T(x_0)$ *tels que*

$$\langle Aw_0, \eta(x,x_0)\rangle \in -int\, C(x_0), \text{ pour tout } x \in K.$$

REFERENCES

1. Q. H. Ansari, *On generalized vector variational-like inequalities,* Ann. Sci. Math. Québec **19** (1995), 131–137.
2. G. Y. Chen, *Existence of solutions for a vector variational inequality: an extension of Hartmann-Stampacchia theorem,* J. Optim. Theory Appl. **74** (1992), 445–456.
3. G. Y. Chen and G. M. Cheng, *Vector Variational Znequality and Vector Optimization*, Lecture Notes in Econom. and Math. Systems, vol. 285, Spnnger-Verlag, Berlin, 1987, pp. 408–416.
4. G. Y. Chen and B. D. Craven, *Approximate dual and approximate vector variational inequality for multiobjective optirnizution,* J. Austral. Math. Soc. Ser. A **47** (1989), 418–423.
5. ______, *A vector variational inequality and optirnizution over an efficient set,* Z. Oper. Res. **34** (1990), 1–12.
6. G. Y. Chen and X. Q. Yang, *The vector complementarity problem and its equivalences with the weak minimal element in ordered spaces,* J. Math. Anal. Appl. **153** (1990), 136–158.
7. K. Fan, *A generalization of Tychonoff's fixed-point theorem,* Math. Ann. **142** (1961), 305–310.
8. F. Giannessi, *Theorems of alternative, quadratic programs and complementarity problems* (R. W. Cottle, F. Giannessi and J. L. Lions, ed.), John Wiley and Sons, Chichester, 1980, pp. 151–186.
9. T. C. Huang, *Extended generalized variational inequalities,* Comput. Math. Appl. **26** (1993), 21–30.
10. C. R. Jou and J. C. Yao, *Extension of generalized multi-valued variational inequalities,* Appl. Math. Lett. **6** (1993), 21–25.
11. G. M. Lee, D. S. Kim, B. S. Lee and S. J. Cho, *Generalized vector variational inequality and Fuzzy extension,* Appl. Math. Lett. **6** (1993), 47–51.

12. A. H. Siddiqi, Q. H. Ansari and R. Ahmad, ***On vector variational-like inequalities,*** To appear in Indian J. Pure Appl. Math. (1997).
13. A. H. Siddiqi, Q. H. Ansari and A. Khaiiq, ***On vector variational inequalities,*** J. Optim. Theory Appl. **84** (1995), 171–180.
14. X. Q. Yang, ***Generalized convex functions and vector variational inequalities,*** J. Optim. Theory Appl. **79** (1993), 563–580.

Q. H. ANSARI
DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH – 202 002 (INDIA)
E-MAIL: mmt06qha@amu.nic.in