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ON VARIATIONAL-LIKE INEQUALITIES

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RÉSUMÉ. La théorie des inéquations variationnelles est devenue un important outil pour l'étude d'une grande classe de problèmes issus de diverses branches des mathématiques et de l'ingénierie. Parida, Sahoo et Kumar ont récemment introduit et étudié un problème d'inéquation de type variationnel (variational-like inequality problem) dans $\mathbb{R}^n$. Nous étudions, dans le présent travail, l'existence de solutions à ce problème dans les espaces de Banach réflexifs ainsi que dans les espaces vectoriels topologiques.

ABSTRACT. The theory of variational inequalities has become a powerful tool in studying a wide class of problems arising in various branches of mathematical and engineering sciences. Recently Parida, Sahoo and Kumar introduced and studied a variational-like inequality problem in $\mathbb{R}^n$. In this paper we study the existence of the solution of this problem in reflexive Banach spaces and topological vector spaces.

1. Introduction. Variational inequality theory has become a powerful and effective tool in studying a wide class of problems arising in various branches of mathematical and engineering sciences. In fact, this theory is an eminently applicable branch of mathematics, which contains a wealth of new ideas as well as plenty of inspiration and motivation for further research work. This theory has been extended and generalized in different directions, because of its applications in different branches of science, engineering, optimization, economics, equilibrium theory, etc. The variational-like inequality problem is one of the generalized form of variational inequality problem, which was introduced and studied by Parida, Sahoo and Kumar [5]. They developed a theory for existence of solution of variational-like inequality in $\mathbb{R}^n$ and have shown its relationship with mathematical programming. Motivated and inspired by the research work going in this area, in the present paper we prove the existence of the solution of variational-like inequality problem in reflexive Banach spaces and Hausdorff topological vector spaces.

Let $\langle E, E^* \rangle$ be a dual system of locally convex spaces and $K \subset E$ be a closed convex subset of E. Given two continuous mappings $T : E \to E^*$ and $\eta(\cdot,\cdot) : K \times K \to E$, then we consider the problem of finding $u_0 \in K$ such that

$$\langle T(u_0), \eta(u, u_0) \rangle \geq 0, \text{ for all } u \in K, \tag{1.1}$$

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which is called a variational-like inequality problem, introduced by Parida, Sahoo and Kumar [5].

If $\eta(u, u_0) = u - u_0$, then problem (1.1) reduces to the problem of finding $u_0 \in K$ such that

$$\langle T(u_0), u - u_0 \rangle \geq 0, \text{ for all } u \in K, \tag{1.2}$$

which is called a variational inequality problem.

2. Existence theory in reflexive Banach spaces. Let E be a reflexive Banach space and K be a nonempty closed convex subset of E. We denote by 2^E, the set of all nonempty subsets of E and by conv(A), for all $A \subset E$, the convex hull of A.

We need the following concepts:

Definition 2.1. A mapping $T : K \to E^*$ is said to be η-monotone if there exists a continuous mapping $\eta : K \times K \to E$ such that

$$\langle T(u) - T(v), \eta(u, v) \rangle \geq 0, \text{ for all } u, v \in K.$$

T is called hemicontinuous, if for every $u, v \in K$, the mapping $t \to \langle T(u + tv), v \rangle$ is continuous at 0^+.

Definition 2.2. A mapping $F : E \to 2^E$ is called KKM-map, if for every finite subset $\{u_1, u_2, \dots, u_n\}$ of E, conv$(\{u_1, u_2, \dots, u_n\}) \subset \bigcup_{i=1}^n F(u_i)$.

Lemma 2.1 (Ky Fan [2]**).** *Let A be an arbitrary nonempty set in a topological vector space E and $F : A \to 2^E$ be a KKM-map. If $F(u)$ is closed for all $u \in A$ and is compact for at least one $u \in A$ then*

$$\bigcap_{u \in A} F(u) \neq \varnothing.$$

It is well-known that Minty lemma [4] plays an important role in research in the theory of variational inequalities, we extend this lemma for variational-like inequalities.

Lemma 2.2. *Let $T : K \to E^*$ be η-monotone and hemicontinuous mapping on K. Also, let $\eta : K \times K \to E$ be continuous mapping which is linear in the first argument and $\langle T(u), \eta(u, u) \rangle = 0$ for all $u \in K$. Then u satisfies:*

$$u \in K : \quad \langle T(u), \eta(v, u) \rangle \geq 0, \textit{ for all } v \in K \tag{2.1}$$

if and only if it satisfies

$$u \in K : \quad \langle T(v), \eta(v, u) \rangle \geq 0, \textit{ for all } v \in K. \tag{2.2}$$

Proof. Let $u \in K$ be a solution of (2.1). Since T is η-monotone, for every $v \in K$, we have

$$\langle T(v) - T(u), \eta(v, u) \rangle \geq 0.$$

Thus

$$\langle T(v), \eta(v, u) \rangle \geq 0, \text{ for all } v \in K.$$

Now we show that (2.2) implies (2.1). Let $w \in K$, and set, for $0 < t < 1$, $v = tw + (1-t)u \in K$, because K is convex. Hence by (2.2), for $t > 0$

$$\langle T(u + t(w-u)), \eta(tw + (1-t)u, u)\rangle \geq 0.$$

Since η is linear in the first argument and $\langle T(u), \eta(u,u)\rangle = 0$, for all $u \in K$, we have

$$\langle T(u + t(w-u)), t\eta(w,u)\rangle \geq 0.$$

Dividing by t, we have

$$\langle T(u + t(w-u)), \eta(w,u)\rangle \geq 0.$$

Since T is hemicontinuous at 0, we may allow $t \to 0$, we obtain

$$\langle T(u), \eta(w,u)\rangle \geq 0, \text{ for any } w \in K.$$

Hence the lemma is proved. □

Theorem 2.1. *Let E be a reflexive Banach space and let K be a nonempty bounded closed convex subset of E such that*

1°. *$T : K \to E^*$ is η-monotone and hemicontinuous mapping on K,*
2°. *$\eta : K \times K \to E$ is continuous mapping which is linear in the first argument,*
3°. *$\langle T(u), \eta(u,u)\rangle = 0$, for all $u \in K$.*

Then there exists $u_0 \in K$ such that

$$\langle T(u_0), \eta(u, u_0)\rangle \geq 0, \textit{ for all } u \in K.$$

Proof. Let

$$F_1(v) = \{u \in K : \langle T(u), \eta(v,u)\rangle \geq 0\}, \ v \in K. \tag{2.3}$$

First, we prove that F_1 is a KKM-mapping on K. Suppose that $\{u_1, u_2, \ldots, u_n\} \subset K$, $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \geq 0$, $i = 1, \ldots, n$, and

$$u = \sum_{i=1}^n \alpha_i u_i \notin \bigcup_{i=1}^n F_1(u_i).$$

Then we have

$$\langle T(u), \eta(u_i, u)\rangle < 0.$$

Since the set $\{v \in K : \langle T(u), \eta(v,u)\rangle < 0\}$ is convex for every $u \in K$, we have

$$\langle T\Big(\sum_{i=1}^n \alpha_i u_i\Big), \eta\Big(\sum_{i=1}^n \alpha_i u_i, \sum_{i=1}^n \alpha_i u_i\Big)\rangle < 0,$$

which is a contradiction to assumption 3°. So we derived $\mathrm{conv}(\{u_1, u_2, \ldots, u_n\}) \subset \bigcup_{i=1}^n F_1(u_i)$, and therefore F_1 is the KKM-mapping on K. Let

$$F_2(v) = \{u \in K : \langle T(v), \eta(v,u)\rangle \geq 0\}, \ v \in K. \tag{2.4}$$

We have $F_1(v) \subset F_2(v)$, for every $v \in K$. Indeed, let $u \in F_1(v)$, so that

$$\langle T(u), \eta(v,u)\rangle \geq 0.$$

By the η-monotonicity of T, we have

$$\langle T(v), \eta(v,u)\rangle \geq \langle T(u), \eta(v,u)\rangle \geq 0,$$

that is $u \in F_2(v)$.

Thus F_2 is also a KKM-mapping on K. By Lemma 2.2, we have

$$\bigcap_{v\in K} F_1(v) = \bigcap_{v\in K} F_2(v).$$

Moreover, $F_2(v)$, for every $v \in K$, is closed, since T and $\eta(\cdot,\cdot)$ are continuous.

We now equip E with the weak topology. Then K, as a closed bounded convex subset in the reflexive Banach space E, is weakly compact for every $v \in K$. Since $F_2(v) \subset K$ and $F_2(v)$ is closed. Hence $F_2(v)$ is weakly compact. By Lemma 2.1,

$$\bigcap_{v\in K} F_1(v) = \bigcap_{v\in K} F_2(v) \neq \varnothing.$$

Hence there exists $u_0 \in K$ such that

$$\langle T(u_0), \eta(u,u_0)\rangle \geq 0, \text{ for all } u \in K. \quad \square$$

3. Existence theory in Hausdorff topological vector spaces. We will use the following result:

Theorem 3.1 [2]. *Let E be a nonempty compact convex set in a Hausdorff topological vector space and A be a subset of $E \times E$ with the following properties:*

1°. *For each $u \in E$, $(u,u) \in A$.*
2°. *For any fixed $u \in E$, the set $A_u = \{v \in E : (u,v) \in A\}$ is closed in E.*
3°. *For each fixed $v \in E$ the set $A_v = \{u \in E : (u,v) \notin A\}$ is convex.*

Then there exists a point $v_0 \in E$ such that $E \times \{v_0\} \subset A$.

If $\langle E, E^*\rangle$ is a dual system of locally convex spaces, K is a closed convex subset of E and D is a nonempty compact convex subset of K then we have the following result:

The bilinear form $\langle\cdot,\cdot\rangle$ is supposed to be continuous.

Theorem 3.2. *Assume that*

1°. *$T : K \to E^*$ is continuous,*
2°. *$\eta : K \times K \to E$ is continuous and linear in the first argument, and*
3°. *$\langle T(u), \eta(u,u)\rangle \geq 0$, for all $u \in D$.*

Then there exists $u_0 \in D \subset K$ such that

$$\langle T(u_0), \eta(u,u_0)\rangle \geq 0, \textit{ for all } u \in D \subset K.$$

Proof. Let
$$A = \{(u,v) \in D \times D : \langle T(v), \eta(u,v)\rangle \geq 0\}. \tag{3.1}$$
The theorem will be proved if we show that assumptions (1°), (2°) and (3°) of Theorem 3.1 are satisfied.

Indeed, for every $u \in D$, $(u,u) \in A$ if and only if
$$\langle T(u), \eta(u,u)\rangle \geq 0,$$
but from assumptions this inequality is satisfied.

Clearly, from the continuity of T and $\eta(\cdot,\cdot)$, we have that for every $u \in D$, the set
$$A_u = \{v \in D : \langle T(v), \eta(u,v)\rangle \geq 0\}$$
is closed. Now to complete the proof of the theorem we show that for every $v \in D$, the set
$$B_v = \{u \in D : (u,v) \notin A\}$$
is convex.

Let $u_1, u_2 \in B_v$ and $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha + \beta = 1$, we have that
$$\langle T(v), \eta(\alpha u_1, v)\rangle < 0 \tag{3.2}$$
and
$$\langle T(v), \eta(\beta u_2, v)\rangle < 0 \tag{3.3}$$
which implies that
$$\langle T(v), \eta(\alpha u_1 + \beta u_2, v)\rangle < 0,$$
that is $\alpha u_1 + \beta u_2 \in B_v$. Hence B_v is convex.

Now from Theorem 3.1 there exists $u_0 \in D$ such that $D \times \{u_0\} \subset A$ which implies that
$$\langle T(u_0), \eta(u, u_0)\rangle \geq 0, \text{ for every } u \in D,$$
and the proof is finished. □

Theorem 3.3. *Assume that*

(1°) $T : K \to E^*$ *is continuous,*
(2°) $\eta(\cdot,\cdot) : K \times K \to E$ *is continuous,*
(3°) *there exists a real valued function* $h : K \times K \to \mathbb{R}$ *such that*
 (i) $\langle T(v), \eta(u,v)\rangle + h(u,v) \geq 0$, *for every* $(u,v) \in K \times K$,
 (ii) *the set* $\{u \in K : h(u,v) > 0\}$ *is convex for every* $v \in K$,
 (iii) $h(u,u) \leq 0$, *for every* $u \in K$,
 (iv) *there exists a nonempty compact convex subset* D *of* K *such that for every* $v \in K \setminus D$ *there exists a point* $u \in D$ *with*
$$\langle T(v), \eta(u,v)\rangle < 0.$$

Then the variational-like inequality (1.1) *has a solution.*

Proof. For each element $u \in K$, we denote

$$D(u) = \{v \in D : \langle T(v), \eta(u,v)\rangle \geq 0\}$$

and from assumptions (1°) and (2°) we have $D(u)$ is closed in D.

We know that every element $u_0 \in \bigcap_{u \in K} D(u)$ is a solution of the problem (1.1), we prove that $\bigcap_{u \in K} D(u) \neq \varnothing$.

Since D is compact it is sufficient to show that the family $\{D(u)\}_{u \in K}$ has a finite intersection property.

Let $u_1, u_2, \ldots, u_m \in K$ be given.

We put $A = \operatorname{conv}\left(D \cup \{u_1, u_2, \ldots, u_m\}\right)$ and we have that A is a compact convex subset of K.

We consider the following multivalued mappings:

$$F_1(u) = \{v \in A : \langle T(v), \eta(u,v)\rangle \geq 0\}, \text{ for every } u \in K$$

and

$$F_2(u) = \{v \in A : h(u,v) \leq 0\}, \text{ for every } u \in K.$$

Since the bilinear form $\langle \cdot, \cdot \rangle$ is continuous, from assumptions (1°) and (2°), it follows that $F_1(u)$ is closed subset of a compact convex set A. Hence $F_1(u)$ is compact. Also from assumptions 3°(i) and 3°(iii), $F_1(u)$ is nonempty.

Now we prove that F_2 is a KKM-mapping.

Indeed, if we suppose that there exists $x_1, x_2, \ldots, x_n \in A$ and $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that

$$\sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{j=1}^{n} F_2(x_j)$$

then we have

$$h\left(x_j, \sum_{i=1}^{n} \alpha_i x_i\right) > 0, \text{ for } 1 \leq j \leq n.$$

By assumption 3°(ii), we have

$$h\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \alpha_i x_i\right) > 0$$

which is a contradiction to assumption 3°(iii). Therefore F_2 is a KKM-mapping.

Since from assumption 3°(i), we have $F_2(u) \subset F_1(u)$, for every $u \in K$, we obtain that F_1 is also a KKM-mapping.

Applying Lemma 2.1 to F_1 we get $\bigcap_{u \in A} F_1(u) \neq \varnothing$, that is, there exists a point $v_0 \in A$ such that

$$\langle T(v_0), \eta(u, v_0)\rangle \geq 0, \text{ for all } u \in A.$$

By assumption 3°(iv), we have that $v_0 \in D$ and moreover $v_0 \in D(u_i)$, for every $1 \leq i \leq m$.

Hence $\{D(u)\}_{u \in K}$ has a finite intersection property and the proof is finished. □

Corollary 3.1. *Assume that assumptions* (1°), (2°) *and* 3°(iv) *of Theorem 3.3 are satisfied. Also suppose that*

$$\langle T(u), \eta(u,u)\rangle \geq 0, \textit{ for all } u \in K.$$

Then the variational-like inequality problem (1.1) *has a solution.*

Proof. If we consider $h(u,v) = -\langle T(v), \eta(u,v)\rangle$, then all the assumptions of Theorem 3.3 are satisfied. □

4. Existence theory without convexity. In this section, we prove an existence theorem for variational-like inequalities, by replacing convexity assumptions with merely topological properties.

We need the following concepts:

Definition 4.1. An H-space is a pair $(E, \{\Gamma_A\})$, where E is a topological space and $\{\Gamma_A\}$ is a given family of nonempty contractible subsets of E, indexed by the finite subsets of E such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

A subset D of E is called H-convex, if for every finite subset A of D, it follows that $\Gamma_A \subset D$.

A subset D of E is called weakly H-convex if for every finite subset A of D, it results that $\Gamma_A \cap D$ is nonempty and contractible. This is equivalent to saying that the pair $(D, \{\Gamma_A \cap D\})$ is an H-space.

A subset K of E is called H-compact if for every finite subset A of E, there exists a compact, weakly H-convex set D of E such that $K \cup A \subset D$.

Definition 4.2. Let $(E, \{\Gamma_A\})$ be an H-space. A multivalued mapping $F : E \to E$ is called H-KKM if $\Gamma_A \subset \bigcup_{u\in A} F(u)$, for every finite subset A of E.

Theorem 4.1 [2]. *Let $(E, \{\Gamma_A\})$ be an H-space and $F : E \to E$ an H-KKM multivalued mapping such that*

(a°) *for each $u \in E$, $F(u)$ is compactly closed, that is $B \cap F(u)$ is closed in B for every compact set $B \subset E$,*

(b°) *there exists a compact set $L \subset E$ and an H-compact set $K \subset E$ such that, for every weakly H-convex set D with $K \subset D \subset E$, we have*

$$\bigcap_{u\in D} \big(F(u) \cap D\big) \subset L.$$

Then

$$\bigcap_{u\in E} F(u) \neq \varnothing.$$

Now we have the following:

Theorem 4.2. *Let $(E, \{\Gamma_A\})$ be an H-Banach space and $T : E \to E^*$ and $\eta(\cdot,\cdot) : E \times E \to E$ be continuous. Assume that*

1°. *for each $v \in E$, $B_v = \{u \in E : \langle T(v), \eta(u,v)\rangle < 0\}$ is H-convex or empty,*

2°. *$\langle T(u), \eta(u,u)\rangle \geq 0$, for all $u \in E$,* 3°. *there exists a compact set* $L \subset E$ *and an* H*-compact set* $K \subset E$ *such that for every weakly* H*-convex set* D *with* $K \subset D \subset E$

$$\{v \in D : \langle T(v), \eta(u, v)\rangle \geq 0, \textit{ for all } u \in D\} \subset L.$$

Then there exists $u_0 \in E$ *such that*

$$\langle T(u_0), \eta(u, u_0)\rangle \geq 0, \textit{ for all } u \in E.$$

Proof. Let

$$F(u) = \{v \in E : \langle T(v), \eta(u, v)\rangle \geq 0\}, \ u \in E.$$

We prove that F is a H-KKM mapping. Suppose that F is not an H-KKM mapping. Then there exists a finite subset A of E such that

$$\Gamma_A \not\subset \bigcup_{u \in A} F(u).$$

Thus there exists $z \in \Gamma_A$ such that

$$z \notin F(u), \text{ for all } u \in A$$

that is

$$\langle T(z), \eta(u, z)\rangle < 0, \text{ for all } u \in A.$$

By assumption (1°), $A \subset B_z$ and $\Gamma_A \subset B_z$, since B_z is H-convex. Therefore $z \in B_z$, that is

$$\langle T(z), \eta(z, z)\rangle < 0,$$

which contradicts assumption (2°). Thus $\Gamma_A \subset \bigcup_{u \in A} F(u)$ for every finite subset A of E. Hence F is an H-KKM mapping.

Since T, $\eta(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ are continuous we have $F(u)$ is closed for every $u \in E$, that is condition (a°) of Theorem 4.1 holds. It is easy to see that the assumption (3°) of this theorem and condition (b°) of Theorem 4.1 are the same. Thus we have

$$\bigcap_{u \in E} F(u) \neq \varnothing.$$

Hence there exists $u_0 \in E$ such that

$$\langle T(u_0), \eta(u, u_0)\rangle \geq 0, \text{ for all } u \in E. \quad \square$$

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Résumé substantiel en français. Soit $\langle E, E^* \rangle$ un couple d'espaces vectoriels localement convexes en dualité et $K \subset E$ un sous-espace fermé convexe de E. Nous considérons le problème de trouver un $u_0 \in K$ tel que

$$\langle T(u_0), \eta(u, u_0) \rangle \geq 0, \text{ pour tout } u \in K, \tag{1.1}$$

où $T : E \to E^*$ et $\eta(\cdot, \cdot) : K \times K \to E$ sont des applications données. Ce problème est appelé ici *problème d'inéquation de type variationnel* suite aux travaux de Parida, Sahoo et Kumar [5] qui l'ont considéré dans le cas de $\mathbb{R}^n$.

La motivation principale du présent travail est d'étudier l'existence d'une solution à ce problème dans le contexte des espaces de Banach réflexifs et des espaces vectoriels topologiques.

Si $\eta(u, u_0) = u - u_0$, alors le problème (1.1) se réduit à celui de trouver $u_0 \in K$ tel que

$$\langle T(u_0), u - u_0 \rangle \geq 0, \text{ pour tout } u \in K, \tag{1.2}$$

appelé ici *problème d'inéquation variationnelle.*

En utilisant le lemme 2.1 de Fan et en faisant appel aux notions de η-monotonie et d'hémicontinuité (voir définition 2.1), nous démontrons d'abord l'existence d'une solution à (1.1) dans le contexte des espaces de Banach réflexifs.

Théorème 2.1. *Soit E un espace de Banach réflexif et K un sous-espace fermé convexe borné non vide de E tel que*

1°. $T : K \to E^*$ *est une application η-monotone et hémicontinue,*

2°. $\eta : K \times K \to E$ *est une application continue, linéaire en la première composante,*

3°. $\langle T(u), \eta(u, u) \rangle = 0$, *pour tout $u \in K$.*

Alors il existe $u_0 \in K$ tel que $\langle T(u_0), \eta(u, u_0) \rangle \geq 0$, pour tout $u \in K$.

Dans la troisième section du travail, nous établissons des théorèmes d'existence dans le contexte des espaces vectoriels topologiques Hausdorff.

Soit $\langle E, E^* \rangle$ un couple d'espaces vectoriels localement convexes en dualité, $K \subset E$ un sous-espace fermé convexe de E et D un compact non vide convexe de K. Nous démontrons les résultats suivants (la forme bilinéaire $\langle \cdot, \cdot \rangle$ étant supposée continue).

Théorème 3.2. *Supposons que*

1°. $T : K \to E^*$ *est continue,*

2°. $\eta : K \times K \to E$ *est continue, linéaire en la première composante,*

3°. $\langle T(u), \eta(u, u) \rangle \geq 0$, *pour tout $u \in D$.*

Alors il existe $u_0 \in D \subset K$ tel que $\langle T(u_0), \eta(u, u_0) \rangle \geq 0$, pour tout $u \in D \subset K$.

Théorème 3.3. *Supposons que*

1°. $T : K \to E^*$ *est continue,*

2°. $\eta : K \times K \to E$ *est continue,*

3°. *il existe une fonction à valeurs réelles $h : K \times K \to \mathbb{R}$ telle que*

(i) $\langle T(v), \eta(u, v) \rangle + h(u, v) \geq 0$, *pour tout $(u, v) \in K \times K$,*

(ii) *l'ensemble $\{u \in K : h(u, v) > 0\}$ est convexe pour tout $v \in K$,*

(iii) $h(u,u) \leq 0$, *pour tout* $u \in K$,
(iv) *il existe un sous-ensemble compact convexe non vide* D *de* K *tel que pour tout* $v \in K \setminus D$ *il existe un point* $u \in D$ *satisfaisant* $\langle T(v), \eta(u,v)\rangle < 0$.

Alors le problème d'inéquation de type variationnel (1.1) *possède une solution.*

Dans la dernière section de l'article, nous faisons appel aux concepts de H-espace de Banach, de H-compacité et de H-(presque) convexité (voir définition 4.1) pour énoncer et démontrer un autre théorème d'existence pour lequel les hypothèses de convexité sont remplacées par des propriétés purement topologiques.

Théorème 4.2. *Soit* $(E, \{\Gamma_A\})$ *un* H*-espace de Banach,* $T : E \to E^*$ *et* $\eta : E \times E \to E$ *des applications continues. Supposons que*

1°. *pour tout* $v \in E$, $B_v = \{u \in E : \langle T(v), \eta(u,v)\rangle < 0\}$ *est* H*-convexe ou vide,*
2°. $\langle T(u), \eta(u,u)\rangle \geq 0$, *pour tout* $u \in E$,
3°. *il existe un ensemble compact* $L \subset E$ *et un ensemble* H*-compact* $K \subset E$ *tels que pour tout ensemble presque* H*-convexe* D *satisfaisant* $K \subset D \subset E$ *on ait*

$$\{v \in D : \langle T(v), \eta(u,v)\rangle \geq 0, \text{ pour tout } u \in D\} \subset L.$$

Alors il existe $u_0 \in E$ *tel que* $\langle T(u_0), \eta(u, u_0)\rangle \geq 0$, *pour tout* $u \in E$.

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