Appendix II Introduction to Matrices

Objectives: The purpose of covering material in the Appendix is to quickly gain familiarity with notions and methods of linear algebra which are needed to understand then study of systems of linear first order differential equations.

We will cover following three parts

Part 1: Basic definitions and theory of matrices

Part 2: Solving linear systems and finding matrix inverse using elementary row operations

Part 3: Eigenvalues and eigenvectors

- A lot of material in the appendix is already known to you from previous courses.
- Such material will be covered quickly using slides.

Review of elementary stuff about matrices

Matrix: $m \times n$ matrix is a collection of numbers or functions in m rows and n columns

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} \qquad (*)$$

You are required to review the following basic notions from your prep-year notes

- Size of a matrix ($m \times n$?)
- Equal matrices
- Zero matrix
- Square matrix
- Diagonal matrix
- Identity matrix
- Row vector ($1 \times n$ matrix)
- Column vector ($n \times 1$ matrix)
- Addition of matrices (when is it possible and how to do it)
- Multiplication of a matrix by a number

Will be discussed in class, if really needed.

Matrix operations

1) Matrix addition [self review]

2) Multiplication of a matrix by a number [self review]

3) Multiplication of matrices

AB defined only if
the number of columns of A = number of rows of B.

How: The $(i,j)th$ element of AB is obtained by multiplying the corresponding elements of row 'i' of A and column 'j' of B and adding the product.

$$\underset{A}{\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ a_{41} & a_{42} & a_{43} \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}} \underset{B}{\begin{bmatrix} \bullet & \bullet & b_{13} & \bullet & \bullet \\ \bullet & \bullet & b_{23} & \bullet & \bullet \\ \bullet & \bullet & b_{33} & \bullet & \bullet \end{bmatrix}} = \underset{C}{\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & c_{43} & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}}$$

$$c_{43} = a_{41}b_{13} + a_{42}b_{23} + a_{43}b_{33}$$

Example: Find the missing entry in the following

$$\begin{bmatrix} 2 & 1 & -3 \\ 5 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 2 \\ 3 & 1 & -2 & 4 \\ 0 & -1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 5 & ? & -11 & 5 \\ 11 & -2 & 31 & 22 \end{bmatrix}$$

Note:
$size(A) = 2\times 3$, $size(B) = 3\times 4$,
$size(C) = 2\times 4$ (Why?)

Transpose of a matrix

The transpose of the $m\times n$ matrix (*) is the $n\times m$ matrix A^T given by

$$A^T = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

In other words, the rows of a matrix A become the columns of its transpose.

Multiplicative identity matrix 'I'

A square matrix I such that

$$AI = IA = A$$

for any matrix A

Example 3×3 identity matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Finding determinant of a square matrix

- 1×1 matrix $\quad A=[a];\quad \det A = a$
- 2×2 matrix $\quad A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix};\quad \det A = a_{11}a_{22} - a_{12}a_{21}$

Example:

- $A=\begin{bmatrix} 5 & 4 \\ 1 & 4 \end{bmatrix};\quad \det A = 16$
- $A=\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix};\quad \det A = 4$
-

Determinants of higher order matrices

- Calculated as row or column expansion
- We understand with the help of examples.

See Example 2, 3 done in class

Non-Singular matrix

a matrix with non-zero determinant

Inverse of a square matrix

A square matrix A is called invertible if there exists a matrix B such that

$$AB = BA = I.$$

B is called inverse of A and is denoted as A^{-1}.

only non-singular matrices have inverse

- We will find inverse using row operations.
- See part 2 of Appendix.

Differentiation & integration of matrices whose entries are functions

- While studying systems of differential equations we will be dealing with matrices whose entries are variables or functions.
- Also our solutions will be in the form of matrices and vectors.
- Hence we need to learn how to differentiate & integrate such matrices.

To differentiate or integrate a matrix whose entries are functions

just

differentiate or integrate each entry

See Examples 4, 5 done in class

Formula for the Inverse of a Matrix:
Let A be an $n \times n$ nonsingular matrix and let $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column from A. Then

$$A^{-1} = \frac{1}{\det A}(C_{ij})^T .$$

Augmented matrix of a linear system:

A way of compactly recording the essential information of a linear system in the form a matrix as explained below.

Example: Given a linear system $\begin{aligned} x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \end{aligned}$

- We can associate a **coefficient matrix** with it as $\begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$
- If we also include the constants on the right of system, we get the **augmented matrix** associated to the system $\begin{bmatrix} 1 & -2 & -1 \\ 4 & 5 & 3 \end{bmatrix}$.

In general, the augmented matrix for a system
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n &= b_m \end{aligned}$$

is $$(A \mid B) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots\cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} & b_m \end{array}\right].$$

Exercise: Give the augmented matrix for
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_1 + 4x_2 + 6x_3 &= -9 \end{aligned}$$

How can we use augmented matrix to solve a linear system?

Example 3.2.1. {To demonstrate that elimination method can be performed in parallel on augmented matrix to solve linear system}

To solve
$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 2 \\ x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + x_2 + 6x_3 &= 26 \end{aligned}$$
or
$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 2 \\ 1 & 2 & 3 & 1 \\ 4 & 1 & 6 & 26 \end{array}\right]$$

Solution:

→ **Step I:**

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 3x_2 + 2x_3 &= 2 \\ 4x_1 + x_2 + 6x_3 &= 26 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 2 & 2 \\ 4 & 1 & 6 & 26 \end{array}\right] \quad \text{SWAP}(R_1, R_2)$$

→ **Step II:**

$$\begin{aligned} & & x_1 + 2x_2 + 3x_3 &= 1 \\ &\text{Eq2+}(-2)\text{Eq1} & -x_2 - 4x_3 &= 0 \\ &\text{Eq3+}(-4)\text{Eq1} & -7x_2 - 6x_3 &= 22 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & 0 \\ 0 & -7 & -6 & 22 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

→ **Step III:**

$$\begin{aligned} & & x_1 + 2x_2 + 3x_3 &= 1 \\ & & -x_2 - 4x_3 &= 0 \\ &\text{Eq3+}(-7)\text{Eq2} & 22x_3 &= 22 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 22 & 22 \end{array}\right] \quad R_3 - 7R_2$$

→ **Step IV:**

- 3^{rd} equation implies $\boxed{x_3 = 1}$
- Back substitution of x_3 in 2^{nd} equation gives $\boxed{x_2 = -4}$
- Back substitution of x_2, x_3 in 1^{st} equation gives $\boxed{x_1 = 6}$

Solution by back substitution

→ **Step V:** Unique solution $x_1 = 6,\ x_2 = -4,\ x_3 = 1$

→

Message: By performing operations similar to elementary operations of elimination method we can convert augmented matrix into a nice form which can easily solve the system.

Q.1. Operations for matrices (similar to operations of elimination method)?

Q.2. Nice form of augmented matrix?

Elementary Row Operations for Matrices

<u>Just an adaptation of elementary operation of last topic</u>

	Operations for elimination method	Corresponding operations for matrices
1	Interchange two equations	Interchange two rows
2	Multiply any equation by a non-zero constant	Multiply any row by a non-zero constant
3	Replace one equation by "the sum of itself and a multiple of another equation"	Replace one row by "the sum of itself and a multiple of another row"

Elementary row operations

The elementary row operations, when applied to augmented matrix of a linear system, do not change the solution of the system. {Can you see why?}

Notation as on previous page

- Now that we know the operations that can be performed on augmented matrix of a system. Let's come to the second question posed above.
- What is the "nice form" in which we want to convert the augmented matrix, in order to solve the system easily?
    - Answer: Echelon form (described below).

What is echelon form of a matrix?

A matrix is in echelon form if

1. All zero-rows (all elements zero) are below non-zero rows.
2. The first non-zero element of a row lies on the right of the first non-zero element of the previous row.
3. All elements below the first non-zero element of each row are zero.

Example: Which of the following are in echelon form?

(i) $\begin{bmatrix} -2 & 6 & 0 & 7 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & 2 & 9 \\ 0 & 7 & -3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 7 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 7 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$

Terminology for matrices in echelon form:

- The first non-zero element of each row is called its leading entry.
- A column containing a leading entry is called a leading column
- In case of augmented matrices, the variable corresponding to leading columns are called leading variables and all others are called free variables.

Can you see the leading entries and leading columns in above echelon matrices?

Leading (or free) variables will play important role in writing solutions (see later part of this topic).

How to convert a matrix into echelon form

Use elementary row operations on the matrix in a systematic way as explained in the examples below. The process is called Gaussian elimination method.

Gaussian elimination method for converting a matrix to echelon form

- Begin with left most non-zero column (it's your first leading column)
- Choose the leading entry & bring at the top of current leading column (how? Of course by interchanging rows)
- Convert the entries below the leading entry into zeros.
- Jump to the next lower level
- Repeat the above steps until echelon form is achieved.

Example 3.2.2

Transform the matrix $\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$ into echelon form. Indicate the leading entries and leading columns.

Solution: [Done in class]

Calculation tips

- Finish the calculations of whole row and then go to next row.
- Avoid fractional calculations as much as possible (by interchanging rows).

Exercise:

Transform the matrix $\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$ into echelon form. Indicate the leading entries and corresponding leading columns.

Hint: you will need to change rows to avoid fractional calculations.

Formula for the Inverse of a Matrix: Let A be an $n \times n$ nonsingular matrix and let $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the *i*th row and *j*th column from *A*. Then

$$A^{-1} = \frac{1}{\det A} (C_{ij})^T .$$

$$(A \mid I) = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

$$(A \mid I) \quad \rightarrow \quad (I \mid A^{-1})$$

- Perform row operations on *A* until *I* is obtained. This means that *A* is nonsingular
- By simultaneously applying the same row operations to *I*, we get A^{-1}

Definition: A vector X is called an ***eigen vector*** of the matrix A if it satisfies the equation

$$AX = \lambda X \qquad (*)$$

In appropriate form above equation can be written as:

$$(A - \lambda I)X = 0 \qquad (*)$$

In the above equation I is the identity matrix of the order equal to that of matrix A and λ is called the ***eigen value***.

- The non-zero vector X satisfying (*) has solution if

$$\text{del}(A - \lambda I) = 0 \qquad (**)$$

- If matrix A and identity matrix I are $n \times n$ matrices, then equation (**) is a polynomial of degree n in λ given by

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \cdots + a_{n-1}\lambda + a_n = 0$$

  This polynomial equation is called a ***characteristic equation***. Its solutions are called ***Eigen Values***.

* Illustration on Finding Eigen Values and Eigen Vector

Find Eigen values for the matrix given by $A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$

Step I. Write the Eigen vector $x = \begin{pmatrix} x \\ y \end{pmatrix}$

Step II. The Eigen value problem (3) for this matrix becomes:

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$

Step III. Re-write the above equation in the form:

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (4)$$

$$\begin{pmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Step IV. Evaluation of Eigen Values

The above system has a solution if

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0 = \lambda^2 - 7\lambda + 6$$

The characteristic equation is a polynomial of degree 2 above and its two roots (Eigen values) are $\lambda_1=1$ and $\lambda_2=6$.

Step V. Determination of Eigen Vectors

Substitute the Eigen values one by one in (4).

(Va): $\lambda_1=1$.

Equation (4) takes the form:

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(Vb): $\lambda_2=6$

Equation (4) takes the form:

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \beta \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

* Note

Suppose we are interested in finding the Eigen vectors of some 2x2 matrix whose characteristic equation is of the form:

$$\lambda^2+a=0,$$

where "a" is some positive real number. The above equation has two roots given by **λ=+i, -i**. Such roots, as you know, are called complex roots and always appear in ordered pairs of each other. In this situation we can easily find that their corresponding Eigen vectors (if they exist) are also complex conjugate of each other.

∗ **Note**

The Eigen vectors v_1, v_2, …,v_n associated with distinct Eigen values λ_1, λ_2, λ_3,…, λ_n are linearly independent.

∗ **Example** Find Eigen values and Eigen vectors of the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Step I. Eigen Value Equation

$$(A-\lambda I)x = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Step II. Characteristic Equation

$$|A-\lambda I| = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$$

Step III. Eigen Vectors corresponding to λ=i, and λ=-i

For $\lambda = i$ *, Equation* $(A - \lambda A)x$ *gives*

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow RREF \ \ of \ \ the \ \ matrix \ \ becomes \ : \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow$$

$$y = t \ and \ \ x = -it \ \ giving \ \ \vec{x} = t\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$For \ \ \lambda = -i \ we \ \ similarly \ \ get : \vec{x} = t\begin{pmatrix} i \\ 1 \end{pmatrix}$$

∗ **Note**

It can also be possible that a characteristic equation may have repeated (twice) real roots. Then what one needs to do is to find two linearly independent Eigen vectors. A method for fining such vectors will be shown through one of the exercise given below.