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To my wife Rosy

Contents

Chapter 1

Metric Spaces

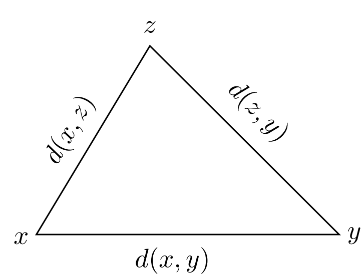
1.1 Definition and Examples of Metric Spaces

Definition 1.1. Let X be a nonempty set. A real-valued function d defined on $X \times X$ is said to be a *metric* on X if it satisfies the following conditions:

(M1) $d(x, y) = 0$ if and only if $x = y$,
(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$, and (symmetry)
(M3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. (triangle inequality)

The set X together with a given metric d on X is called a *metric space* and is denoted by (X, d). If there is no confusion likely to occur we, sometimes, denote the metric space (X, d) by X.

The triangle inequality may be interpreted as that "the length of one side of a triangle can not exceed the sum of the length of the other two sides". Equivalently, the distance from x to y via any intermediate point z can not be shorter than the direct distance from x to y.



The metric d has the following properties:

1. *For all $x, z \in X$, $d(x, z)$ is always nonnegative:*
   If we put $y = x$ in the triangle inequality (M3), we get
$$d(x, x) \leq d(x, z) + d(z, x).$$
By using (M1) and (M2), we obtain
$$2d(x, z) \geq 0 \text{ and hence } d(x, z) \geq 0.$$

2. Since the distances are generally greater going via an additional point, then they are greater going via any number of additional points $z_1, z_2, \ldots, z_n$; from the triangle inequality (M3), it follows by induction that for any $x, y, z_1, z_2, \ldots, z_n \in X$,

$$\begin{aligned} d(x,y) &\leq d(x,z_1) + d(z_1,y) \\ &\leq d(x,z_1) + d(z_1,z_2) + d(z_2,y) \\ &\leq d(x,z_1) + d(z_1,z_2) + d(z_2,z_3) + d(z_3,y) \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ &\leq d(x,z_1) + d(z_1,z_2) + \cdots + d(z_n,y). \end{aligned}$$

3. *For any $x, y, z \in X$, we have*

$$|d(x,z) - d(y,z)| \leq d(x,y).$$

For it follows from (M2) and (M3) that

$$d(x,z) \leq d(x,y) + d(y,z)$$

and

$$d(y,z) \leq d(y,x) + d(x,z) = d(x,y) + d(x,z).$$

Thus

$$-d(x,y) \leq d(x,z) - d(y,z) \leq d(x,y).$$

Problem 1.1. *If $x_1, x_2, y_1, y_2 \in X$, then prove that*

$$|d(x_1,y_1) - d(x_2,y_2)| \leq d(x_1,x_2) + d(y_1,y_2).$$

Examples of Metric Spaces

Example 1.1. Let $X = \mathbb{R}$, the set of all real numbers. For $x, y \in X$, define

$$d(x,y) = |x - y|.$$

Then (X, d) is a metric space and the metric d is called the *usual metric on* $\mathbb{R}$.

Verification. For all $x, y, z \in X$, we have

(M1) $d(x,y) = |x - y| = 0$ if and only if $x = y$;

(M2) $d(x,y) = |x - y| = |-(x - y)| = |y - x| = d(y,x)$;

(M3) $$\begin{aligned} d(x,y) &= |x - y| = |(x - z) + (z - y)| \\ &\leq |x - z| + |z - y| \\ &= d(x,z) + d(z,y). \end{aligned}$$

In the verification of (M3) in Example 1.1, we used the fact that $|a+b| \leq |a| + |b|$ for real numbers a and b. This property is also true for complex numbers a and b. Hence, we have the following example:

Example 1.2. Let $X = \mathbb{C}$, the set of all complex numbers. For $x, y \in X$, define

$$d(x, y) = |x - y|.$$

Then (X, d) is a metric space and the metric d is called the *usual metric on* $\mathbb{C}$.

Example 1.3. Let X be any nonempty set. For $x, y \in X$, define

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

Then (X, d) is a metric space. The metric d is called *discrete metric* and the space (X, d) is called *discrete metric space*.

Remark 1.1. It shows that on each nonempty set, we can always define at least one metric, called discrete metric.

Example 1.4. Let $X = \mathbb{R}^2$, the set of all points in the coordinate plane. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in X define

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Then (X, d) is a metric space and $d(x, y)$ is the natural distance between two points in a plane.

Verification. For any $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ in X.

(M1)
$$\begin{aligned} d(x,y) = 0 \quad &\Leftrightarrow \quad \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0 \\ &\Leftrightarrow \quad (x_1 - y_1)^2 = 0 \text{ and } (x_2 - y_2)^2 = 0 \\ &\Leftrightarrow \quad x_1 = y_1 \text{ and } x_2 = y_2 \\ &\Leftrightarrow \quad x = y. \end{aligned}$$

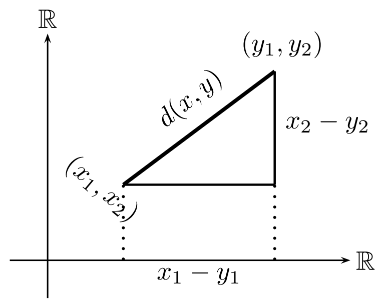
(M2)
$$\begin{aligned} d(x,y) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \\ &= d(x, y). \end{aligned}$$

(M3)
$$\begin{aligned}[d(x,y)]^2 &= (x_1-y_1)^2+(x_2-y_2)^2\\ &= [(x_1-z_1)+(z_1-y_1)]^2+[(x_2-z_2)+(z_2-y_2)]^2\\ &= (x_1-z_1)^2+(z_1-y_1)^2\\ &\quad +2\left[\underbrace{(x_1-z_1)}_{a}\underbrace{(z_1-y_1)}_{b}+\underbrace{(x_2-z_2)}_{c}\underbrace{(z_2-y_2)}_{d}\right]\\ &\quad +(x_2-z_2)^2+(z_{2-}y_2)^2\end{aligned}$$

Taking $x_1-z_1=a$, $z_1-y_1=b$, $x_2-z_2=c$ and $z_2-y_2=d$, and since

$$(ab+cd)^2 \le (a^2+c^2)(b^2+d^2),$$

we have



$$\begin{aligned}[d(x,y)]^2 &\le (x_1-z_1)^2+(x_2-z_2)^2\\ &\quad +2\sqrt{(x_1-z_1)^2+(x_2-z_2)^2}\sqrt{(z_1-y_1)^2+(z_2-y_2)^2}\\ &\quad +(z_1-y_1)^2+(z_2-y_2)^2\\ &= [d(x,z)]^2+2d(x,z)\,d(z,y)+[d(z,y)]^2\\ &= [d(x,z)+d(z,y)]^2.\end{aligned}$$

Therefore,

$$d(x,y) \le d(x,z)+d(z,y).$$

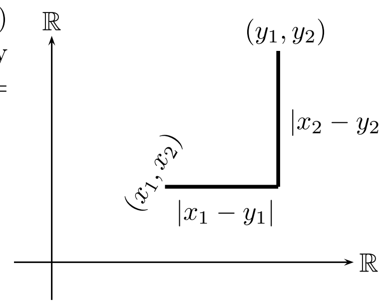
Example 1.5. Let $X=\mathbb{R}^2$. For $x=(x_1,x_2)$, $y=(y_1,y_2)$ in X, define

$$d(x,y)=|x_1-y_1|+|x_2-y_2|.$$

Then (X,d) is a metric space.

Verification. The conditions (M1) and (M2) are obvious. We prove only condition (M3). For $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ in X,

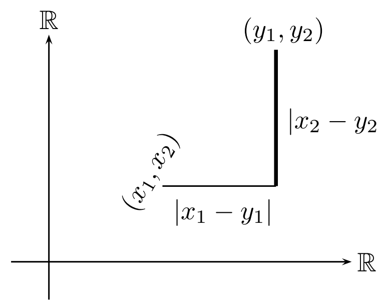
$$\begin{aligned} d(x,y) &= |x_1 - y_1| + |x_2 - y_2| \\ &\leq |x_1 - z_1| + |z_1 - y_1| \\ &\quad + |x_2 - z_2| + |z_2 - y_2| \\ &= d(x,z) + d(z,y). \end{aligned}$$



Example 1.6. Let $X = \mathbb{R}^2$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in X, define

$$d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

Then (X, d) is a metric space.



Remark 1.2. Examples 1.4, 1.5 and 1.6 show that more than one metric can always be defined on a nonempty set.

Problem 1.2. *Let $X = \mathbb{R}^2$ and $x = (0,2)$, $y = (3,6)$ in X. Find the distance between x and y by using the metrics of Examples 1.4, 1.5 and 1.6.*

Example 1.7. Let $X = \mathbb{R}^n$, the set of all ordered n-tuples of real numbers. For $x = (x_1, x_2, \ldots, x_n) \in X$ and $y = (y_1, y_2, \ldots, y_n) \in X$, we define

(a) $\quad d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ (called usual metric)

(b) $\quad d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}, \qquad p \geq 1$ (called taxicab metric)

(c) $\quad d_\infty(x,y) = \max_{1 \leq i \leq n} \{|x_i - y_i|\}.$ (called max metric)

Verification. In view of Examples 1.4, 1.5 and 1.6, it is easy to verify that d_1, d_p and d_∞ are metrics on X.

The triangular inequality (M3) in the case of d_p requires the use of

Minkowski inequality[1]

$$\begin{aligned} d_p(x,y) &= \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |x_i - z_i + z_i + y_i|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{i=1}^{n} |x_i - z_i| + |z_i - y_i|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{i=1}^{n} |x_i - z_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |z_i - y_i|^p\right)^{\frac{1}{p}} \\ &= d_p(x,z) + d(x,y). \end{aligned}$$

Remark 1.3. Let $X = \mathbb{C}^n$, the set of all n-tuples of complex numbers. We can define the metrics d_1, d_p and d_∞ on X in the same way as in Example 1.7.

Example 1.8. Let ℓ^∞ be the space of all bounded sequences of real or complex numbers, that is,

$$\ell^\infty = \left\{ \{x_n\} \subseteq \mathbb{R} \text{ or } \mathbb{C} : \sup_{1 \le n < \infty} |x_n| < \infty \right\}.$$

For $x = \{x_n\} \in \ell^\infty$ and $y = \{y_n\} \in \ell^\infty$, define

$$d_\infty(x,y) = \sup_{1 \le n < \infty} |x_n - y_n|.$$

Then it is easy to verify that d_∞ is a metric on ℓ^∞ and (d_∞, ℓ^∞) is a metric space.

Example 1.9. Let s be the space of all sequences of real or complex numbers, that is,

$$s = \{\{x_n\} \subseteq \mathbb{R} \text{ or } \mathbb{C}\}.$$

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[1] MINKOWSKI INEQUALITY: Let $1 \le p < \infty$. If $x_i, y_i \in \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) $(i = 1, 2, \cdots, n)$, then $\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$.
Let $0 < p \le 1$. If $x_i, y_i \in \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) $(i = 1, 2, \cdots, n)$, then

$$\sum_{i=1}^{n} |x_i + y_i|^p \le \sum_{i=1}^{n} |x_i|^p + \sum_{i=1}^{n} |y_i|^p.$$

For $x = \{x_n\}$ and $y = \{y_n\}$ in s, define

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}.$$

Then (s, d) is a metric space.

Verification. The series $\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$ is convergent since its ith term is less than $\frac{1}{2^i}$. The conditions (M1) and (M2) can be easily verified.

Let $x = \{x_i\}$, $y = \{y_i\}$ and $z = \{z_i\}$ in s. Then by triangular inequality, we have

$$|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$$

and hence[2]

$$\begin{aligned} \frac{|x_i - y_i|}{1 + |x_i - y_i|} &\leq \frac{|x_i - z_i| + |z_i - y_i|}{1 + |x_i - y_i| + |z_i - y_i|} \\ &= \frac{|x_i - z_i|}{1 + |x_i - z_i| + |z_i - y_i|} + \frac{|z_i - y_i|}{1 + |x_i - z_i| + |z_i - y_i|} \\ &\leq \frac{|x_i - z_i|}{1 + |x_i - z_i|} + \frac{|z_i - y_i|}{1 + |z_i - y_i|}. \end{aligned}$$

Multiplying both sides by $\frac{1}{2^i}$ and summing with respect to i, we obtain

$$d(x,y) \leq d(x,z) + d(z,y).$$

Problem 1.3. *Let c be the space of all convergent sequences of real or complex numbers. For $x = \{x_n\}$ and $y = \{y_n\}$ in c, define*

$$d(x,y) = \sup_{1 \leq i < \infty} |x_i - y_i|.$$

Then prove that d is a metric on c and (c, d) is a metric space.

Example 1.10. Let $1 \leq p < \infty$. Consider the space ℓ^p of all sequences $\{x_n\}$ of real or complex numbers such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Let $x = \{x_n\}$ and $y = \{y_n\} \in \ell^p$, we define

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}.$$

Then, (ℓ^p, d) is a metric space.

[2]Let $0 \leq \alpha \leq \beta$. Then $\alpha + \alpha\beta \leq \beta + \alpha\beta$. Dividing both sides by $(1+\alpha)(1+\beta)$, we have

$$\frac{\alpha}{1+\alpha} \leq \frac{\beta}{1+\beta}.$$

Verification. The conditions (M1) and (M2) can be easily verified. Let $x = \{x_n\}$, $y = \{y_n\}$ and $z = \{z_n\}$ be sequences in ℓ^p. Then

$$\begin{aligned} d(x,y) &= \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^{\infty} |x_n - z_n + z_n - y_n|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{n=1}^{\infty} |x_n - z_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |z_n - y_n|^p\right)^{\frac{1}{p}} \\ &\quad \text{(by Minkowski's inequality)} \\ &= d(x,z) + d(z,y). \end{aligned}$$

Example 1.11. Let $B[a,b]$ be the space of all bounded real-valued functions defined on $[a,b]$, that is, $B[a,b] = \{f : [a,b] \to \mathbb{R} : f(t) \le k$ for all $t \in [a,b]\}$. For $f, g \in B[a,b]$, we define

$$d(f,g) = \sup_{t\in[a,b]} |f(t) - g(t)|.$$

Then $(B[a,b], d)$ is a metric space.

Example 1.12. Let $C[a,b]$ be the space of all continuous real-valued functions defined on $[a,b]$. For $f, g \in C[a,b]$, we define the following metrics on $C[a,b]$:

$$d_\infty(f,g) = \max_{t\in[a,b]} |f(t) - g(t)|$$

and

$$d_1(x,y) = \int_a^b |f(t) - g(t)|\, dt,$$

where the integral is the Riemann integral which is possible because the functions f and g are continuous on $[a,b]$. Then d_∞ and d_1 are metrics on $C[a,b]$.

The metric d_∞ measures the "distance" from f to g as the maximum of vertical distances from points $(t, f(t))$ to $(t, g(t))$ on the graphs of f and g, respectively. $d_1(f,g)$ represents as a measure of the distance between the functions f and g to be the area enclosed between their graphs from $x = a$ to $x = b$.

We can also define an another metric on $C[a,b]$. Let $f, g \in C[a,b]$, define

$$d(f,g) = \left(\int_a^b |f(t) - g(t)|^p \, dt \right)^{\frac{1}{p}} \quad \text{for } p \geq 1.$$

Then $(C[a,b], d)$ is a metric space.

Problem 1.4. *Let (X,d) be a metric space. Then prove that*

(a) $|d(x,z) - d(z,y)| \leq d(x,y)$ *for all* $x, y \in X$*;*
(b) $|d(x_1,y_1) - d(x_2,y_2)| \leq d(x_1,x_2) + d(y_1,y_2)$ *for all* $x_1, x_2, y_1, y_2 \in X$*.*

Problem 1.5. *Let $\mathbb{K}$ be the set of all real or complex numbers. Prove that for each $x, y \in \mathbb{K}$,*

$$d_1(x,y) = \min\{1, |x-y|\}$$

and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ |x| + |y|, & \text{if } x \neq y \end{cases}$$

are metrics on $\mathbb{K}$.

Problem 1.6. *Let $X = [0,1)$. For each $x, y \in X$, we define*

$$d(x,y) = |x-y|.$$

Prove that (X,d) is a metric space.

Problem 1.7. *Let $X = \mathbb{Q}$, the set of all rational numbers. Show that for each $x, y \in X$,*

$$d(x,y) = |x-y|$$

is a metric on X.

Problem 1.8. *Let $X = \mathbb{R}^2$ and for each $x = (x_1,x_2)$, $y = (y_1,y_2) \in X$, let*

$$d(x,y) = \begin{cases} |x_1 - y_1|, & \text{if } x_2 = y_2 \\ |x_1| + |y_1| + |x_2 - y_2|, & \text{if } x_2 \neq y_2. \end{cases}$$

Then prove that (X,d) is a metric space.

Problem 1.9. *Let $X = \mathbb{R}^2$ and $x = (0,2)$, $y = (3,6)$ in X. Then find the distance between x and y by using the metrics (a) usual, (b) taxicab, (c) max, and (d) discrete.*

Problem 1.10. *Let d_1 and d_2 are metrics on a set X. Is $\min\{d_1, d_2\}$ also a metric on X? Justify your answer.*

Problem 1.11. *Let (X_i, d_i), $i = 1, 2, \ldots, n$, be metric spaces and $X = X_1 \times X_2 \times \cdots \times X_n$. Then prove that for each $x = (x_1, x_2, \ldots, x_n) \in X$ and $y = (y_1, y_2, \ldots, y_n) \in X$,*

$$d_1(x, y) = \max_{1 \le i \le n} d_i(x_i, y_i)$$

and

$$d_2(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i)$$

are metrics on X.

Problem 1.12. *Let (X, d) be a metric space. Prove that for each $x, y \in X$,*

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X.

$\left(\text{Hint: } \textit{Use } \frac{a}{1+a} + \frac{b}{1+b} \ge \frac{a+b}{1+a+b} \textit{ for all } a \ge 0,\ b \ge 0. \right)$

Problem 1.13. *Let $X = c$, the space of all convergent sequences $\{x_n\}$, where $\lim\limits_{n\to\infty} x_n$ exists and finite, and for each $x = (x_1, x_2, \ldots) \in X$ and $y = (y_1, y_2, \ldots) \in X$, let*

$$d(x, y) = \sup_i |x_i - y_i|.$$

Then prove that (X, d) is a metric space.

1.2 Distance Between Sets and Diameter of a Set

Definition 1.2. Let (X, d) be a metric space and let A and B be nonempty subsets of X. The *distance between the sets A and B*, denoted by $\rho\,(A, B)$, is given by

$$\rho\,(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$$

Since $d(x, y) = d(y, x)$, we have $\rho\,(A, B) = \rho\,(B, A)$.

If A consists of a single point x, then

$$\rho\,(\{x\}, B) = \inf\{d(x, y) : y \in B\}.$$

It is called the *distance of a point $x \in X$ from the set B*, and is denoted by $\rho\,(x, B)$.

Remark 1.4. (i) The equation $\rho(x, B) = 0$ does not imply that x belongs to B.
(ii) If $\rho(A, B) = 0$, then it does not imply that A and B have common points.

Example 1.13. Let $A = \{x \in \mathbb{R} : x > 0\}$ and $B = \{x \in \mathbb{R} : x < 0\}$ be subsets of $\mathbb{R}$ with the usual metric. Then $\rho(A, B) = 0$, but A and B have no common point. If $x = 0$ then $\rho(x, B) = 0$, but $x \notin B$.

Definition 1.3. Let (X, d) be a metric space and let A be a nonempty subset of X. The *diameter of* A, denoted by $\delta(A)$, is given by

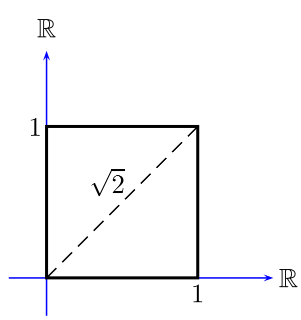
$$\delta(A) = \sup\{d(x, y) : x, y \in A\}.$$

The set A is called *bounded* if $\delta(A) \leq k < \infty$. In other words, A is bounded if its diameter is finite, otherwise it is called *unbounded*.

In particular, the metric space (X, d) is bounded if the set X is bounded.

Example 1.14. (a) The real line with the usual metric is an unbounded metric space.
(b) In $\mathbb{R}$ with the usual metric, the intervals $[a, b]$, (a, b), $[a, b)$ and $(a, b]$ are bounded. But $[a, \infty)$ and $(-\infty, a]$ are not bounded.
(c) The space s of all sequences of real or complex numbers with the metric defined in Example 1.9 is a bounded space since $d(x, y) < \sum_{i=1}^{n} \frac{1}{2i}$.
(d) Every set in a discrete metric space (X, d) is bounded and its diameter is 1.

Example 1.15. Consider the unit sphere $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ in $\mathbb{R}^2$. With the usual metric d, the diameter of S is $\sqrt{2}$; with the taxicab metric, its diameter is 2; with the max metric, its diameter is 1; and with the discrete metric its diameter is 1.



Remark 1.5. Let (X, d) be a metric space. We can define other metrics on X with the help of d in the following manner:

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)} \quad \text{and} \quad d_2(x, y) = \min\{1, d(x, y)\}.$$

Then d_1 and d_2 are metrics on X and with these metrics (X, d_1) and (X, d_2) are bounded metric spaces irrespective of whether the metric space (X, d) is bounded or not.

Problem 1.14. *Determine the distance from $(3, 4)$ to the unit square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$ with respect to the metrics (a) usual, (b) taxicab, (c) max, and (d) discrete.*

Problem 1.15. *Let $A = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$ be the unit sphere in $\mathbb{R}^3$. Compute the diameter of A with respect to each of the following metrics: (a) usual, (b) taxicab, (c) max, and (d) discrete.*

Problem 1.16. *If (X, d) is a metric space with discrete metric and A is a subset of X with at least two elements, then show that the diameter of A is 1.*

Problem 1.17. *Let $A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ and let $x = (1, 1)$. Find the distance from x to A for the following metrics: (a) usual, (b) taxicab, (c) max, and (d) discrete.*

Problem 1.18. *Let A and B be nonempty subsets of a metric space (X, d). Prove that*

(a) $\delta(A) = 0$ *if and only if A is a singleton set;*
(b) For each $x \in A$, $y \in B$, $\rho(A, B) \leq d(x, y)$;
(c) If $A \subseteq B$, *then* $\delta(A) \leq \delta(B)$;
(d) For each $x \in A$, $y \in B$, $d(x, y) \leq \delta(A \cup B)$;
(e) $\delta(A \cup B) \leq \delta(A) + \rho(A, B) + \delta(B)$;
(f) If $A \cap B \neq \emptyset$, *then* $\delta(A \cup B) \leq \delta(A) + \delta(B)$;
(g) $d(x, A) \leq d(x, y) + d(y, A)$ *for all* $x, y \in X$.

Proof. [Proof of (e) and (f)]. Let a and b be arbitrary elements of A and B, respectively, and let $x, y \in A \cup B$. If both x and y are in A, then $d(x, y) \leq \delta(A)$. If both x and y are in B, then $d(x, y) \leq \delta(B)$.

If $x \in A$ and $y \in B$, then by the triangle inequality, we have

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, y) \\ &\leq d(x, a) + d(a, b) + d(b, y) \\ &\leq \delta(A) + d(a, b) + \delta(B). \end{aligned}$$

Similarly, if $x \in B$ and $y \in A$, we have

$$d(x,y) \leq \delta(A) + d(a,b) + \delta(B).$$

Thus,

$$d(x,y) \leq \delta(A) + d(a,b) + \delta(B) \quad \text{for all } x, y \in A \cup B.$$

Therefore,

$$\delta(A \cup B) \leq \delta(A) + d(a,b) + \delta(B) \quad \text{for all } x \in A,\ y \in B.$$

Hence,

$$\delta(A \cup B) \leq \delta(A) + \rho(A,B) + \delta(B).$$

Now, if $A \cap B \neq \emptyset$, we have $\rho(A,B) = 0$ and hence $\delta(A \cup B) \leq \delta(A) + \delta(B)$. □

1.3 Open Sets and Interior Points

Definition 1.4. Let (X,d) be a metric space. Given a point $x_0 \in X$ and a real number $r > 0$, the sets

$$S_r(x_0) = \{y \in X : d(x_0, y) < r\}$$

and

$$S_r[x_0] = \{y \in X : d(x_0, y) \leq r\}$$

are called *open sphere* (or *open ball*) and *closed sphere* (or *closed ball*), respectively, with center x and radius r.

Remark 1.6. (a) The open and closed spheres are always nonempty, since $x_0 \in S_r(x_0) \subseteq S_r[x_0]$.
(b) Every open (respectively, closed) sphere in $\mathbb{R}$ with the usual metric is an open (respectively, closed) interval. But the converse is not true; for example, $(-\infty, \infty)$ (respectively, $[-\infty, \infty]$) is an open (respectively, closed) interval in $\mathbb{R}$ but not an open (respectively, closed) sphere.

Example 1.16. 1. In the metric space $\mathbb{R}$ with the usual metric, the spheres $S_r(x_0)$ and $S_r[x_0]$ are intervals

$$(x_0 - r, x_0 + r) \quad \text{and} \quad [x_0 - r, x_0 + r],$$

respectively.

2. In the metric space $\mathbb{C}$ with the usual metric, the sphere $S_r(z_0)$ and $S_r[z_0]$ are circular discs

$$|z - z_0| < r \quad \text{and} \quad |z - z_0| \leq r,$$

respectively, where $z_0 \in \mathbb{C}$ and $r > 0$.
3. Let X be a nonempty set with the discrete metric d. Then the open sphere $S_r(x_0)$ is

$$S_r(x_0) = \begin{cases} \{x_0\}, & \text{if } 0 < r \leq 1, \\ X, & \text{if } r > 1, \end{cases}$$

and the closed sphere $S_r[x_0]$ is

$$S_r[x_0] = \begin{cases} \{x_0\} & \text{if } 0 < r < 1, \\ X, & \text{if } r \geq 1. \end{cases}$$

4. Let $X = [0, 1)$ be a metric space with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$. Then the open sphere $S_r(0)$ is

$$S_r(0) = \begin{cases} [0, r), & \text{if } r \leq 1, \\ [0, 1), & \text{if } r > 1, \end{cases}$$

and the closed sphere $S_r[0]$ is

$$S_r[0] = \begin{cases} [0, r), & \text{if } r < 1, \\ [0, 1), & \text{if } r \geq 1. \end{cases}$$

5. In $\mathbb{R}^2$, the open sphere with center 0 and radius 1 with respect to the metrics d_1, d_2 and d_∞, respectively, (in Example 1.7), are

$$S_1^1(0) = \left\{ y = (y_1, y_2) \in \mathbb{R}^2 : |y_1| + |y_2| < 1 \right\},$$

$$S_1^2(0) = \left\{ y = (y_1, y_2) \in \mathbb{R}^2 : |y_1|^2 + |y_2|^2 < 1 \right\},$$

and

$$S_1^\infty(0) = \left\{ y \in (y_1, y_2) \in \mathbb{R}^2 : \max(|y_1|, |y_2|) < 1 \right\}.$$

Similarly, we can define the closed spheres.
6. In the metric space $C[a, b]$, the open sphere $S_r(f_0)$ with center f_0 and radius r is the set of continuous functions g such that

$$\sup_{t \in [0,1]} |f(t) - g(t)| < r,$$

that is, the set of continuous functions g whose graphs lie within the shaded band of vertical width $2r$ centered on the graph of f_0.

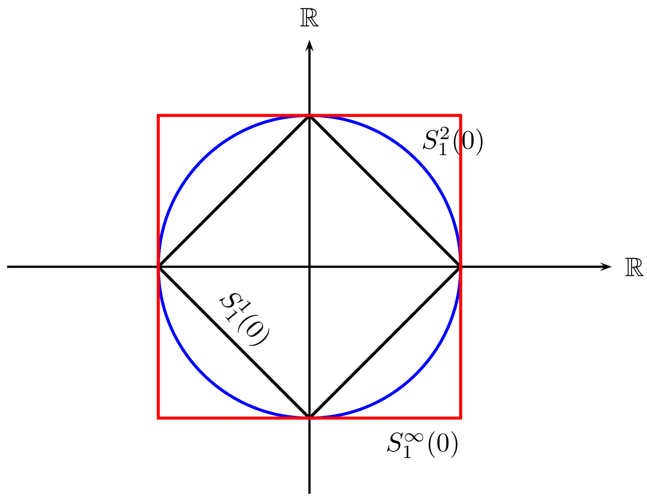


Fig. 1.1 Balls in $\mathbb{R}^2$

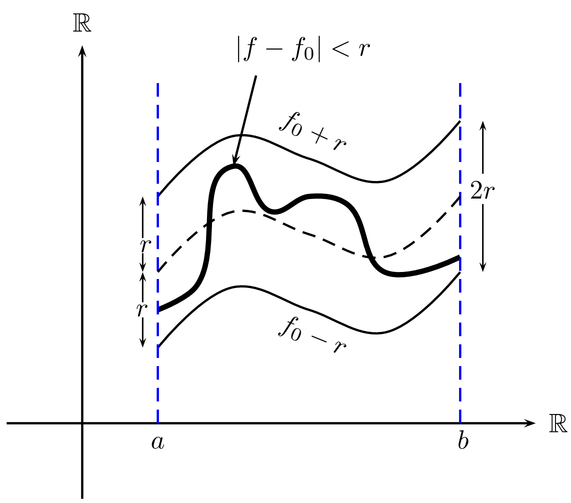
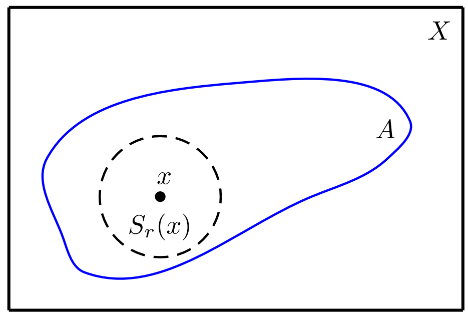


Fig. 1.2 Ball in $C[a, b]$

Definition 1.5. Let A be a nonempty subset of a metric space X.

(i) A point $x \in A$ is said to be an *interior point* of A if x is the center of some sphere contained in A;
In other words, $x \in A$ is an interior point of A if there exists $r > 0$ such

that $S_r(x) \subseteq A$.



(ii) The set of all interior points of A is called *interior* of A and is denoted by A°, that is,

$$A^\circ = \{x \in A : S_r(x) \subseteq A \text{ for some } r > 0\}.$$

(iii) The set A is said to be *open* if each of its points is the center of some open sphere contained entirely in A; That is to say, A is an open set if for each $x \in A$, there exists $r > 0$ such that $S_r(x) \subseteq A$.

(iv) Let $x \in X$. The set A is said to be a *neighbourhood* of x if there exists an open sphere centered at x and contained in A, that is, if $S_r(x) \subseteq A$, for some $r > 0$. In case, A is an open set, it is called an *open neighbourhood* of x.

Remark 1.7. (a) In particular, an open sphere $S_r(x)$ with center x and radius r is a neighborhood of x.
(b) The interior of A is the neighbourhood of each of its points.
(c) Every open set is the neighbourhood of each of its points.
(d) The set A is open if and only if each of its points is an interior point, that is, $A = A^\circ$.

Example 1.17. 1. Let $\mathbb{R}$ be the usual metric space and A be a subset of $\mathbb{R}$.

(a) $A = (a, b)$, $[a, b)$, $[a, b,]$, or $(a, b]$, then $A^\circ = (a, b)$
(b) If $A = \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or the set of irrational numbers, then $A^\circ = \emptyset$.
(c) If A is a finite set, then $A^\circ = \emptyset$.
(d) If $A = C$, the cantor set, then $A^\circ = \emptyset$.
(e) If $A = \emptyset$, then $A^\circ = \emptyset$.
(f) If $A = \mathbb{R}$, then $A^\circ = \mathbb{R}$.

2. Let A be a nonempty subset of a discrete metric space X. Then $A^\circ = A$.

Example 1.18. 1. In $\mathbb{R}$ with the usual metric

(a) $\mathbb{R}$ is an open set;
(b) (a, b) is an open set;
(c) $(a, b]$, $[a, b)$ and $[a, b]$ are not open sets;
(d) The set $\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$ is not open;
(e) A set consists a singleton is not an open set;
(f) The set of all rational numbers $\mathbb{Q}$ is not open. But it is open with respect to the metric $d(x, y) = |x - y|$ defined on $\mathbb{Q}$;
(g) The cantor set C is not an open set.

2. Let $X = [0, 1)$ with the metric $d(x, y) = |x - y|$ for all $x, y \in X$. Then $[0, \alpha)$, $\alpha \leq 1$, is an open set.
3. In the discrete metric space X, every subset of X is an open set.

Remark 1.8. (a) In a metric space X, the empty set $\emptyset$ and the whole space X are open sets.
(b) Whether a set is open or not open depends upon the space in which it is considered. For example, identify the real line $\mathbb{R}$ with horizontal axis $\{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ in $\mathbb{R}^2$. $\mathbb{R}$ is not an open subset of $\mathbb{R}^2$ since $\mathbb{R}$ does not contain any open sphere in $\mathbb{R}^2$.

Theorem 1.1. *Let A and B be two subsets of a metric space X. Then*

(i) $A \subseteq B$ *implies* $A^\circ \subseteq B^\circ$*;*
(ii) $(A \cap B)^\circ = A^\circ \cap B^\circ$*;*
(iii) $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$*.*

Proof. (i) Let $x \in A^\circ$. Then there exists an open sphere $S_r(x) \subseteq A$. Since $A \subseteq B$, $S_r(x) \subseteq B$ and hence $x \in B^\circ$. Thus $A^\circ \subseteq B^\circ$.

(ii) Let $x \in (A \cap B)^\circ$. Then there exists an open sphere $S_r(x) \subseteq A \cap B$. Therefore, $S_r(x) \subseteq A$ and $S_r(x) \subseteq B$ and hence $x \in A^\circ$ and $x \in B^\circ$. So, $x \in A^\circ \cap B^\circ$ and thus $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.

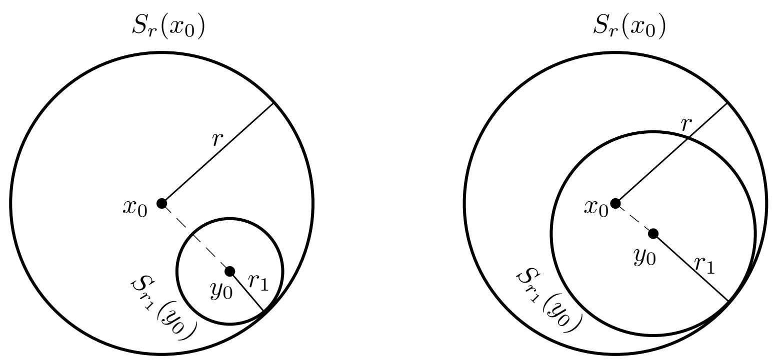
To prove the reverse inclusion, let us suppose that $y \in A^\circ \cap B^\circ$. Then $y \in A^\circ$ and $y \in B^\circ$ and therefore, there exist open spheres $S_{r_1}(y) \subseteq A$ and $S_{r_2}(y) \subseteq B$. Set $r = \min\{r_1, r_2\}$. Then $S_r(y) \subseteq A \cap B$ and hence $y \in (A \cap B)^\circ$. Consequently, $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$.

(iii) Let $x \in A^\circ \cup B^\circ$. Then either $x \in A^\circ$ or $x \in B^\circ$. This implies that there exists an open sphere $S_r(x) \subseteq A$ or $S_r(x) \subseteq B$ for some r. So, we have $S_r(x) \subseteq A \cup B$ and therefore $x \in (A \cup B)^\circ$. Hence $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$. $\square$

Remark 1.9. $(A \cup B)^{\circ} \not\subseteq A^{\circ} \cup B^{\circ}$. For example, let $X = \mathbb{R}$ be the usual metric space and $A = [0,1]$ and $B = [1,2]$. Then $A \cup B = [0,2]$. Note that $A^{\circ} = (0,1)$, $B^{\circ} = (1,2)$ and $(A \cup B)^{\circ} = (0,2)$. This shows that $A^{\circ} \cup B^{\circ} \subseteq (A \cup B)^{\circ}$ but $(A \cup B)^{\circ} \not\subseteq A^{\circ} \cup B^{\circ}$.

Theorem 1.2. *Let (X,d) be a metric space. Then*

(i) each open sphere in X is an open set;
(ii) a subset A of X is open if and only if it is the union of open spheres.



$r_1 < d(x_0, y_0)$ $r_1 > d(x_0, y_0)$

Fig. 1.3

Proof. (i) Let $S_r(x_0) = \{x \in X : d(x, x_0) < r\}$ be an open sphere in X and let $y_0 \in S_r(x_0)$. We have to produce an open sphere centered at y_0 and contained in $S_r(x_0)$. Since $y_0 \in S_r(x_0)$, we have $d(x_0, y_0) < r$. Set

$$r_1 = r - d(x_0, y_0) > 0.$$

Consider

$$S_{r_1}(y_0) = \{y \in X : d(y, y_0) < r_1\}.$$

We have to show that $S_{r_1}(y_0) \subseteq S_r(x_0)$. For this, let $y \in S_{r_1}(y_0)$ be arbitrary. Then $d(y, y_0) < r_1$ and therefore

$$\begin{aligned} d(x_0, y) &\leq d(x_0, y_0) + d(y_0, y) \qquad \text{(by triangle inequality)} \\ &< d(x_0, y_0) + r_1 = r. \end{aligned}$$

Thus $y \in S_r(x_0)$ and consequently, $S_{r_1}(y_0) \subseteq S_r(x_0)$.

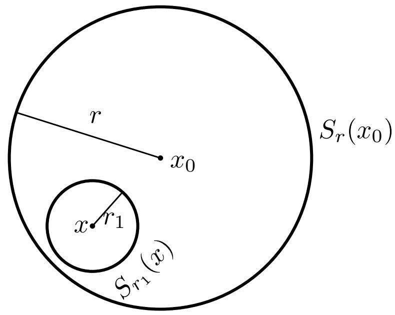


Fig. 1.4

(ii) Suppose that A is an open set. Then, each of its points is the center of an open sphere contained in A. Hence A is the union of all the open spheres contained in it.

To prove the converse part, let us assume that A is the union of a collection $\mathcal{F}$ of open spheres. Let $x \in A$ be arbitrary. Then, x belongs to some open sphere, say $S_r(x_0) \in \mathcal{F}$. Since each open sphere is an open set, x is the center of an open sphere $S_{r_1}(x)$ such that $S_{r_1}(x) \subseteq S_r(x_0)$. But $S_r(x_0) \subseteq A$ and hence $S_{r_1}(x) \subseteq A$. Therefore A is open $\square$

Theorem 1.3. *Let (X, d) be a metric space. Then*

(i) arbitrary union of open sets in X is open;
(ii) finite intersection of open sets in X is open.

Proof. (i) Let Λ be any index set, $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of open sets in X and let $A = \bigcup_{\alpha\in\Lambda} A_\alpha$. Since each A_α is open, it is the union of open spheres for each $\alpha \in \Lambda$. Then A is the union of unions of open spheres. Hence, by Theorem 1.2, A is open.
(ii) Let $\{A_i : i = 1, 2, \ldots, n\}$ be the finite family of open sets in X and let $A = \bigcap_{i=1}^{n} A_i$. Let $x \in A$. Then x is in each A_i. But each A_i is open, hence for each i, there exists $r_i > 0$ such that $S_{r_i}(x) \subseteq A_i$. Set $r = \min\{r_1, r_2, \ldots, r_n\}$. Then

$$S_r(x) \subseteq S_{r_i}(x) \subseteq A_i \quad \text{for each} \quad i = 1, 2, \ldots, n.$$

Therefore $S_r(x) \subseteq \bigcap_{i=1}^{n} A_i = A$ and hence A is open. $\square$

Remark 1.10. Arbitrary intersection of open sets need not be open. For example, let $X = \mathbb{R}$ with the usual metric. Consider the family $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$, of open sets. Then $\bigcap_{i=1}^{\infty} A_n = \{0\}$ which is not open.

Theorem 1.4. *Let A be a subset of a metric space X. Then A° is the largest open subset of A.*

Proof. First of all, we shall prove that A° is an open set. For that, let $x \in A^\circ$ be arbitrary. Then, by definition, there exists an open sphere $S_r(x) \subseteq A$. But $S_r(x)$ is an open set, so each of its points is the center of some open sphere contained in $S_r(x)$. Therefore, each point of $S_r(x)$ is the interior point of A, that is, $S_r(x) \subseteq A^\circ$. Thus, x is the center of an open sphere contained in A°. Hence A° is an open set.

Let $B \subseteq A$ be an arbitrary open set and let $x \in B$. Then there exists $S_r(x) \subseteq B \subseteq A$. This implies that $x \in A^\circ$ and hence $B \subseteq A^\circ \subseteq A$. Since A° is open, A° is the largest open subset of A. □

Remark 1.11. A° is the union of all open subsets of A.

Problem 1.19. *Find the open spheres with center 0 and radius 1 in the metric spaces with respect to the metrics defined in Problems 1.5 and 1.8.*

Problem 1.20. *Let A be a subset of a metric space X. Prove that $(A^\circ)^\circ = A^\circ$.*

Problem 1.21. *In $\mathbb{R}^n$, let R denote the set of points having only rational coordinates and I its complements, that is, the set of points having at least one irrational coordinate. Then prove that $R^\circ = I^\circ = \emptyset$.*

Problem 1.22. *Let (X, d) be a metric space, $a \in X$ and $0 < r < r'$. Prove that the set $\{x \in X : r < d(x, a) < r'\}$ is open in X.*

Problem 1.23. *Let (X, d) be a metric space and*

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that the two metric spaces (X, d) and (X, d^*) have precisely:

(i) the same family of open spheres with one exception. What is this exception?
(ii) the same family of open sets.

Problem 1.24. *Let R be the same as in Problem 1.21. Prove that*

(i) every nonempty open set in $\mathbb{R}^n$ contains a member of R;
(ii) every nonempty open set in $\mathbb{R}^n$ contains infinitely many members of R.

Problem 1.25. *Let (X,d) be a metric space and x, y distinct points of X. Prove that there exist disjoint open spheres centered on x and y.*

1.4 Closed Sets and Closure of Sets

Definition 1.6. Let A be a subset of a metric space X. A point $x \in X$ is called a *limit point* (*accumulation point* or *cluster point*) of A if each open sphere centered on x contains at least one point of A different from x.

In other words, $x \in X$ is a limit point of A if

$$(S_r(x) - \{x\}) \cap A \neq \emptyset.$$

The set of all limit points of A is called *derived set* and it is denoted by A'.

Example 1.19. 1. In the usual metric space $\mathbb{R}$,

(a) if $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$, then $A' = \{0\}$;
(b) if $A = \mathbb{N}$ or $\mathbb{Z}$, then $A' = \emptyset$;
(c) if A is the set of all rational or irrational numbers, then $A' = \mathbb{R}$
(d) every point on the real line is a limit point, and therefore, $\mathbb{R}' = \mathbb{R}$;
(e) if A is a cantor set C, then $A' = C$.

2. If A is a subset of a discrete metric space, then $A' = A$.

Remark 1.12. By the definition of a limit point, we follow that any open sphere centered on a limit point of A must contain infinitely many points of A, that is, to say, a point $x \in X$ is a limit point of A if $S_r(x) \cap A$ is an infinite set for each $r > 0$.

Let $S_r(x)$ contain a point $x_1$of A different from x. If $d(x, x_1) = r_1$, the sphere $S_{r_1}(x)$ contains a point x_2 of A different from x and x_1. And so an indefinitely. It should be noted that a limit point of A is not necessarily a point of A. For example, in Example 1.19 1(a), 0 is the only limit point of the set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$ which is not in A.

In view of the above remark, we have the following definition.

Definition 1.7. A point $x \in X$ is said to be an *isolated point* of X if each open sphere centered on x contains no point of A other than x itself, that is, if $S_r(x) \cap A = \{x\}$ for some $r > 0$.

Remark 1.13. If a point $x \in X$ is not a limit point of A then it is an isolated point. Hence every point of a metric space X is either a limit point or an isolated point of X.

Example 1.20. Consider the metric space $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ with the usual metric given by the absolute value. Then 0 is the only limit point of X while all other points are the isolated point of X.

Definition 1.8. Let A be a subset of a metric space X. The *closure* of A, denoted by $\overline{A}$, is the union of A and the set of all its limit points, that is, $\overline{A} = A \cup A'$.

In other words, $x \in \overline{A}$ if every open sphere $S_r(x)$ with center x and radius $r > 0$ contains a point of A, that is, $x \in \overline{A}$ if $S_r(x) \cap A \neq \emptyset$ for every $r > 0$.

Remark 1.14. Let A and B be subsets of a metric space X. Then

(i) $\overline{\emptyset} = \emptyset$
(ii) $\overline{X} = X$
(iii) $\overline{(\overline{A})} = \overline{A}$
(iv) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$
(v) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
(vi) $\overline{A} = \left(\overline{A}\right)'$
(vii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, but $\overline{A \cap B} \not\supseteq \overline{A} \cap \overline{B}$, for example, in the usual metric space $\mathbb{R}$, consider the sets $A = (0, 1)$ and $B = (1, 2)$. Then $\overline{A} \cap \overline{B} = [0, 1] \cap [1, 2] = \{1\}$, but $\overline{A \cap B} = \emptyset$ and hence $\overline{A \cap B} \not\supseteq \overline{A} \cap \overline{B}$.

Theorem 1.5. *Let (X, d) be metric space and A be a subset of X. Then $x \in \overline{A}$ if and only if $\rho(x, A) = 0$.*

Proof. Since $\rho(x, A) = \inf\{d(x, y) : y \in A\}$, we have $\rho(x, A) = 0$ if and only if every open sphere $S_r(x)$ contains a point of A. Hence $\rho(x, A) = 0$ if and only if $x \in \overline{A}$. □

Definition 1.9. Let A be a subset of a metric space X. The set A is said to be *closed* if it contains all its limit points, that is, $A' \subseteq A$.

It is obvious that A is closed if and only if $\overline{A} = A$.

Example 1.21. In the usual metric space $\mathbb{R}$,

(i) the sets of all rational and irrational numbers are not closed;
(ii) the set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots,\right\}$ is not closed, since $A' = \{0\} \nsubseteq A$;

(iii) the cantor set C is closed since $A' = A \subseteq A$.

Remark 1.15. In a metric space X, every finite set, empty set and whole space are closed sets.

Problem 1.26. *Verify that every subset of the discrete metric space is closed.*

Theorem 1.6. *Let A be a subset of a metric space X. Then, A is closed if and only if the complement of A is an open set.*

Proof. Let A be closed and $x \in A^c$, the complement of A, be arbitrary. Then $x \notin A$ and also x cannot be a limit point of A since A is closed. Then there exists an open sphere $S_r(x)$ such that $S_r(x) \cap A = \emptyset$. This implies that $S_r(x) \subseteq A^c$ for some $r > 0$. Since $x \in A^c$ is arbitrary, each point of A^c is the center of some open sphere which is contained in A^c. Hence A^c is open.

Conversely, assume that A^c is open. Let $x \in X$ be a limit point of A. If $x \in A$, then A contains all its limit points and hence A is closed. If $x \notin A$, then $x \in A^c$. Since A^c is open, there exists an open sphere $S_r(x) \subseteq A^c$. Consequently, $S_r(x) \cap A = \emptyset$ for some $r > 0$. Hence x cannot be a limit point of A which contradicts to our assumption. Therefore $x \in A$. This proves that A is closed. □

Theorem 1.7. *In a metric space (X, d), every closed sphere is a closed set.*

Proof. Let $S_r[x]$ be a closed sphere in X. Then it is sufficient to show that $(S_r[x])^c$, the complement of $S_r[x]$, is an open set. Let $y_1 \in (S_r[x])^c$ be arbitrary. Then $y \notin S_r[x]$ and therefore $d(x, y) > r$.

Set $r_1 = d(x, y) - r > 0$. Let $z \in S_{r_1}(y)$. Then $d(z, y) < r_1$. By triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y)$$

and we have

$$d(x, z) \geq d(x, y) - d(z, y) > d(x, y) - r_1 = r.$$

Therefore $z \notin S_r[x]$ and hence $z \in (S_r[x])^c$. Thus $S_{r_1}(y) \subseteq (S_r[x])^c$. But $y \in (S_r[x])^c$ being arbitrary, each point of $(S_r[x])^c$ is the center of some open sphere contained in $(S_r[x])^c$. Hence $(S_r[x])^c$ is an open set. □

By using De Morgan's law

$$\bigcap_{\alpha \in \Lambda} (A_\alpha^c) = \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right)^c$$

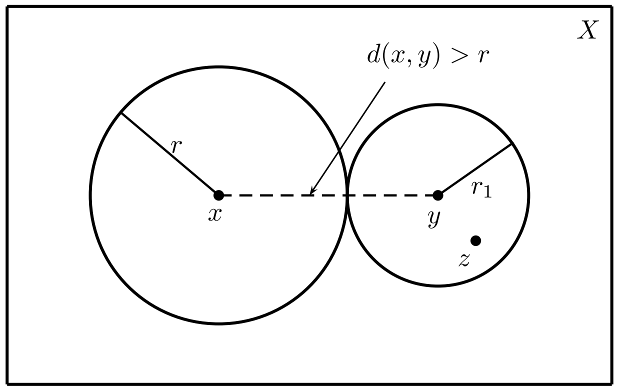


Fig. 1.5

and

$$\bigcup_{i=1}^{n} A_i^c = \left(\bigcap_{i=1}^{n} A_i\right)^c$$

and Theorem 1.3, we have the following result.

Theorem 1.8. *In a metric space X,*

(i) the arbitrary intersection of closed sets in X is closed; and
(ii) the finite union of closed sets in X is closed.

Remark 1.16. The arbitrary union of closed sets need not be closed.

Example 1.22. Consider the family $\left\{\left[\frac{1}{n}, 2\right] : n \in \mathbb{N}\right\}$ of closed sets in the usual metric space $\mathbb{R}$. Then

$$\bigcup \left\{\left[\frac{1}{n}, 2\right] : n \in \mathbb{N}\right\} = (0, 2]$$

which is not a closed set.

Theorem 1.9. *Let (X, d) be a metric space and A be a subset of X. Then $\overline{A}$ is the smallest closed subset of X containing A.*

Proof. Let x be a limit point of $\overline{A}$. Then, for a given $\epsilon > 0$, $\left(S_{\epsilon/2}(x) - \{x\}\right) \cap \overline{A} \neq \emptyset$. This implies that there exists $y \in \overline{A}$ such that $y \in \left(S_{\epsilon/2}(x) - \{x\}\right)$, that is, $d(x, y) < \frac{\epsilon}{2}$. But since $y \in \overline{A}$, we have

$S_{\epsilon/2}(y) \cap A \neq \emptyset$, that is, there exists $z \in A$ such that $z \in S_{\epsilon/2}(y)$. This implies that $d(y,z) < \frac{\epsilon}{2}$. Now, by triangle inequality, we have

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This means that, for every $\epsilon > 0$, the open sphere $S_\epsilon(x)$ contains a point z of A. Hence x is a limit point of A and therefore $x \in \overline{A}$. This proves that $\overline{A}$ is a closed set.

Now, we shall show that $\overline{A}$ is the smallest set containing A. Assume that B is any closed subset of X such that $A \subseteq B$, then it is sufficient to prove that $\overline{A} \subseteq B$. Let $x \in \overline{A}$, then either $x \in A$ or x is a limit point of A. If $x \in A$, then $x \in B$ and hence $\overline{A} \subseteq B$. If x is a limit point of A, then for a given $\epsilon > 0$, $(S_\epsilon(x) - \{x\}) \cap A \neq \emptyset$, that is, there exists a point $y \in A$ such that $y \in \big(S_{\epsilon/2}(x) - \{x\}\big)$. Then $d(x,y) < \epsilon$. But since $A \subseteq B$ and $y \in A$, we have $y \in B$. Therefore, x is a limit point of B. Since B is a closed set, $x \in B$ and thus $\overline{A} \subseteq B$. $\square$

Problem 1.27. *Let A be a subset of a metric space X. Prove that $\overline{A}$ is the intersection of all closed subsets of X containing A.*

Definition 1.10. Let A be a subset of a metric space X. A point $x \in X$ is called a *boundary point* of A if it is neither an interior point of A nor $X \setminus A$, that is, $x \notin A^\circ$ and $x \notin (X \setminus A)^\circ$.

In other words, $x \in X$ is a *boundary point* of A if every open sphere centered on x intersects both A and $X \setminus A$.

The set of all boundary points of A is called the *boundary of* A and it is denoted by $b(A)$.

Example 1.23. 1. Let $\mathbb{R}$ be the usual metric space and $A \subseteq \mathbb{R}$.

(a) If $A = [a,b]$, $[a,b)$, $(a,b]$ or (a,b), then $b(A) = \{a,b\}$.
(b) If $A = \mathbb{N}\,(\text{or } \mathbb{I})$, then $b(A) = \mathbb{N}$ (respectively, I). check?
(c) If $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \right\}$, then $b(A) = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \right\}$.
(d) If $A = \mathbb{Q}$, then $b(A) = \mathbb{R}$. check?
(e) If A is a set of all irrational numbers, then $b(A) = \mathbb{R}$. check?

2. Let (X,d) be a discrete metric space and $A \subseteq X$. Then $b(A) = \emptyset$.

Problem 1.28. *Determine the derive set of the following sets.*

(a) A finite set $A = \{1, 2, \cdots, n\}$. *(Ans. no limit point)*

(b) $R = \{(x_1, x_2) :\in \mathbb{R}^2 : x_1, x_2 \text{ are rational coordinates}\}$
(Ans. entire plane $= \mathbb{R}^2$*.)*

Problem 1.29. *Let A be a subset of a metric space X. Prove that*

(i) $\overline{(X \setminus A)} = X \setminus A^\circ$*, that is,* $\overline{(A^c)} = (A^\circ)^c$.
(ii) $(A^c)^\circ = \left(\overline{A}\right)^c$

Problem 1.30. *Let (X, d) be a metric space and A a closed subset of X. Prove that $x \in A$ if and only if $d(x, A) = 0$ and hence*

$$x \in X \setminus A \text{ if and only if } d(x, A) > 0.$$

Problem 1.31. *Let (X, d) be a metric space, $x \in X$ and $A \subseteq X$ be a nonempty set. Prove that $d(x, A) = 0$ if and only if every neighbourhood of x contains a point of A.*

Problem 1.32. *Let (X, d) be a metric space and $A \subseteq X$ be a nonempty set. Show that $x \in \overline{A}$ if and only if $d(x, A) = 0$.*

Problem 1.33. *Let (X, d) be a metric space and A, B be nonempty subsets of X. Show that $d(x, A) = d(x, B)$ for all $x \in X$ if and only if $\overline{K} = \overline{D}$.*

Problem 1.34. *Let A be a subset of a metric space X. Prove that $\overline{A} = X$ if and only if $(X \setminus A)^\circ = \emptyset$, that is, $(A^c)^\circ = \emptyset$.*

Problem 1.35. *Let (X, d) be a metric space and $A \subseteq X$. Prove the following statements.*

(a) $b(A) = b(X \setminus A) = \overline{A} \cap \overline{(X \setminus A)}$.
(b) $b(A) = \overline{A} \setminus A^\circ = \overline{(X \setminus A)} \setminus (X \setminus A)^\circ$
(c) $X \setminus b(A) = A^\circ \cup (X \setminus A)^\circ$*? Check*
(d) $\overline{A} = A \cup b(A)$
(e) $A^\circ = A \setminus b(A)$
(f) *A is closed if and only if $b(A) \subseteq A$*
(g) *A is open if and only if $A \cap b(A) = \emptyset$.*

1.5 Subspaces

Let (X, d) be a metric space and Y a subset of X. We may convert Y into a metric space by restricting the distance function d to $Y \times Y$. In this manner each subset Y of X can be made a metric space $\left(Y, d_{|_{Y \times Y}}\right)$. On the other hand, we may be given two metric spaces (X, d) and (Y, d'). If Y

is a subset of X, it makes sense to ask whether or not d' is the restriction of d.

Definition 1.11. Let (X, d) be a metric space and Y a subset of X. The *relative metric* d_Y on Y is the restriction of the metric function d on $Y \times Y$, that is,

$$d_Y(x, y) = d(x, y) \quad \text{for all } x, y \in Y.$$

It is easy to see that d_Y is a metric on Y. The space (Y, d_Y) is called the *metric subspace* of the metric space (X, d).

In other words, let (X, d) and (Y, d') be metric spaces. We say that (Y, d') is a subspace of (X, d) if

(i) Y is a subset of X;
(ii) $d' = d_{|_{Y\times Y}}$ restriction of d on $Y \times Y$.

Example 1.24.

(1) Let $\mathbb{R}$ be an usual metric space. If $Y = [0,1], (0,1], [0,1)$ or $(0,1)$ and $d_Y(x, y) = |x-y| = d(x, y)$ for all $x, y \in Y$. Then (Y, d_Y) is a subspace of $(\mathbb{R}, |\cdot|)$.
(2) Let $\mathbb{R}$ be the usual metric space and $\mathbb{Q}$ be the set of rational numbers. Define $d_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by
$$d_{\mathbb{Q}}(x, y) = |x-y| = d(x, y) \quad \text{for all } x, y \in \mathbb{Q}.$$
Then $(\mathbb{Q}, d_{\mathbb{Q}})$ is a subspace of $(\mathbb{R}, |\cdot|)$.
(3) Let I^n (the unit n cube) be the set of all n-tuples $(x_1, x_2, \cdots, x_n)$ of real numbers such that $0 \le x_i \le 1$, for $i = 1, 2, \ldots, n$. Define $d_c : I^n \times I^n \to \mathbb{R}$ by
$$d_c(x, y) = \max_{1\le i\le n} \{|x_i - y_i|\}$$
for all $x = (x_1, x_2, \ldots, x_n) \in I^n$ and $y = (y_1, y_2, \cdots, y_n) \in I^n$. Then (I^n, d_c) is a subspace of $(\mathbb{R}^n, d_\infty)$, where d_∞ is the max metric on $\mathbb{R}^n$, that is, $d_\infty(x, y) = \max\limits_{1\le i\le n} \{|x_i - y_i|\}$, for all $x, y \in \mathbb{R}^n$.
(4) Let S^n (the $n-$ sphere) be the set of all $n+1$-tuples $(x_1, x_2, \ldots, x_{n+1})$ of real numbers such that $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$. Define $d_S : S^n \times S^n \to \mathbb{R}$ by
$$d_S(x, y) = \sqrt{\sum_{i=1}^{n+1} (x_i - y_i)^2} = d_2(x, y),$$

where d_2 is a metric on $\mathbb{R}^n$ defined as $d_2(x,y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$,
for all $x, y \in \mathbb{R}^n$. Then (S^n, d_S) is a subspace of $\left(\mathbb{R}^{n+1}, d_2\right)$.

(5) Let A be the set of all $(n+1)$-tuples $(x_1, x_2, \ldots, x_{n+1})$ of real numbers such that $x_{n+1} = 0$. Define $d_A : A \times A \to \mathbb{R}$ by

$$d_A(x,y) = \max_{1 \le i \le n}\{|x_i - y_i|\} = d_\infty(x,y),$$

for all $x = (x_1, x_2, \ldots, x_n, 0) \in A$ and $y = (y_1, y_2, \ldots, y_n, 0) \in A$, where d_∞ is the max metric on $\mathbb{R}^{n+1}$.
Then (A, d_A) is a subspace of $\left(\mathbb{R}^{n+1}, d_\infty\right)$.

(6) Let $\mathbb{P}[a,b]$ be the set of all polynomials defined on $[a,b]$. Define $d_{\mathbb{P}} : \mathbb{P}[a,b] \times \mathbb{P}[a,b] \to \mathbb{R}$ by

$$d_{\mathbb{P}}(f,g) = \max_{t \in [a,b]} |f(t) - g(t)| = d_\infty(f,g),$$

where d_∞ is the max metric on $C[a,b]$. Then $(\mathbb{P}[a,b], d_{\mathbb{P}})$ is a subspace $(C[a,b], d_\infty)$. But $(\mathbb{P}[a,b], d_{\mathbb{P}})$ is not a subspace of $(C[a,b], d)$, where $d(f,g) = \int_b^a |f(t) - g(t)|\,dt$.

The following lemma can be easily proved.

Lemma 1.1. *Let (Y, d_Y) be a subspace of a metric space (X, d). If $a \in Y$ and $r > 0$, then*

$$S'_r(a) = Y \cap S_r(a),$$

where $S_r(a)$ and $S'_r(a)$ are open spheres in (X, d) and (Y, d_Y), respectively.

Theorem 1.10. *Let (Y, d_Y) be a subspace of a metric space (X, d). Then a subset M of Y is a neighbourhood of a point $y \in Y$ if and only if there is a neighbourhood N of y in (X, d) such that $M = Y \cap N$.*

Proof. Let N be a neighbourhood of a point $y \in Y$ in (X, d) such that $M = Y \cap N$. Then there exists an open sphere $S_r(y)$ such that $S_r(y) \subseteq N$. Since $S'_r(y) = Y \cap S_r(y)$, we have $S'_r(y) \subseteq Y \cap N = M$. Hence M is a neighbourhood of $y \in Y$ in (Y, d_Y).

Conversely, suppose that M is a neighbourhood of y in (Y, d_Y). Then there exists an open sphere $S'_r(y) \subseteq M$. Let $N = M \cup S_r(y)$. Then

$$\begin{aligned} Y \cap N &= Y \cap (M \cup S_r(y)) = (Y \cap M) \cup (Y \cap S_r(y)) \\ &= M \cup S'_r(y) = M, \text{ since } M \subseteq Y \end{aligned}$$

Since $S_r(y) \subseteq N$, N is a neighbourhood of y in (X, d). $\square$

Theorem 1.11. *Let (Y, d_Y) be a subspace of a metric space (X, d) and A a subset of Y. Then*

(i) A is open in Y if and only if there exists an open set G in X such that $A = G \cap Y$;
(ii) A is closed in Y if and only if there exists a closed set F in X such that $A = F \cap Y$.

Proof. (i) Let $S_r(x)$ and $S'_r(x)$ be the same as in Lemma 1.1. Suppose that $A = G \cap Y$ and let $x \in A$ be arbitrary. Then we have to show that x is an interior point of A, that is, $x \in A^\circ$ with respect to d_Y metric.

Since $A = G \cap Y$ and $x \in A$, we have $x \in G$ and $x \in Y$. Since G is open in X, there exists $r > 0$ such that $S_r(x) \subseteq G$. Also, since $x \in Y$, we have

$$S'_r(x) = S_r(x) \cap Y \subseteq G \cap Y = A.$$

It follows that x is an interior point of A as a subset of the metric space (Y, d_Y). Hence $x \in A^\circ$ with respect to d_Y metric and hence A is open in Y.

Conversely, assume that A is an open set in Y and let $x \in A$ be arbitrary. Then there exists an open sphere $S'_{r_x}(x)$ such that $S'_{r_x}(x) \subseteq A$. Now

$$\begin{aligned} A &= \bigcup_{x\in A} S'_{r_x}(x) = \bigcup_{x\in A} \left(S_{r_x}(x) \cap Y\right) = \left(\bigcup_{x\in A} S_{r_x}(x)\right) \cap Y \\ &= G \cap Y, \text{ where } G = \bigcup_{x\in A} S'_{r_x}(x). \end{aligned}$$

But G being an arbitrary union of open spheres in X is an open set in X. Hence $A = G \cap Y$, where G is an open set in X.

(ii) A is closed in $Y \Leftrightarrow Y \setminus A$ is open in Y

$$\begin{aligned} &\Leftrightarrow Y \setminus A = G \cap Y, \quad \text{(by part (i)) where } G \text{ is open in } X \\ &\Leftrightarrow A = Y \setminus (G \cap Y) \\ &\Leftrightarrow A = (X \cap Y) \setminus (G \cap Y) \\ &\Leftrightarrow A = (X \setminus G) \setminus \cap Y \\ &\Leftrightarrow A = F \cap Y, \text{ where } F = X \setminus G \text{ is a closed set in } X. \end{aligned}$$

□

Corollary 1.1. *Let (Y, d_Y) be a subspace of a metric space (X, d) and A a subset of X. Then*

(i) A is open in Y and Y is open in X that A is open in X;

(ii) A is closed in Y and Y is closed in X then A is closed in X.

Theorem 1.12. *Let (Y, d_Y) be a subspace of a metric space (X, d) and A a subset of Y. Then*

(i) $x \in Y$ is a limit point of A in Y if and only if x is a limit point of A in X;
(ii) the closure of A in Y, denoted by $cl_A(Y)$, is $cl_X(A) \cap Y$, where $cl_X(A)$ is the closure of A in X. In other words, $cl_Y(A) = cl_X(A) \cap Y$.

Proof. (i) Let $x \in Y$ be a limit point of A in Y. Then the every open sphere $S'_r(x)$ we have $(S'_r(x) - \{x\}) \cap A \neq \phi$.

For any given $r > 0$ we have

$$\begin{aligned}(S_r(x) - \{x\}) \cap A &= (S'_r(x) \cap Y - \{x\}) \cap A \quad (\text{since } A \subseteq Y) \\ &= (S'_r(x) - \{x\}) \cap A \neq \phi.\end{aligned}$$

It follows that x is a limit point of A in X.

The converse can be established by retracting the above steps.

(ii) Since $cl_X(A)$ is closed in X, by previous theorem, $cl_X(A) \cap Y$ is closed in Y. Since $cl_X(A) \cap Y$ contains A and since $cl_Y(A)$ is the intersection of all closed subsets of Y containing A, we must have

$$cl_Y(A) \subseteq cl_X(A) \cap Y.$$

Further, $cl_Y(A)$ is closed in Y, then $cl_Y(A) = F \cap Y$, where F is a closed set in X. Since $A \subseteq cl_Y(A)$, then F is a closed set in X containing A. Since $cl_Y(A)$ is the intersection of all closed sets containing A, we have

$$cl_Y(A) \subseteq F.$$

Hence $cl_Y(A) \cap Y \subseteq F \cap Y = cl_Y(A)$. □

Chapter 2

Completeness

2.1 Introduction

The concept of a sequence, as studied in real analysis, can be extended without any difficulty to a general metric space, and we shall do so have. We shall also discuss the convergence of a sequence in a metric space.

2.2 Convergent Sequences

Definition 2.1. A sequence s in a set X is a mapping from the set of all natural numbers $\mathbb{N}$ into X. The image under a sequence s of a natural number n will be denoted by x_n and will be referred as nth term of the sequence s.

Definition 2.2. Let (X, d) be a metric space. A sequence of $\{x_n\}$ of points of X is said to be *convergent* if there is a point $x \in X$ such that for each $\varepsilon > 0$, these exists a positive integer N such that

$$d(x_n, x) < \varepsilon \quad \text{for all } n > N.$$

The point $x \in X$ is called the of the sequence $\{x_n\}$.

A sequence which is not convergent is said to be *divergent*.

Since $d(x_n, x) < \varepsilon$ is equivalent to $x_n \in S_\varepsilon(x)$, the definition of convergent sequence can be restated as follows:

A sequence $\{x_n\}$ in a metric space X *converges to a point* $x \in X$ if and only if for each $\varepsilon > 0$, there exists a positive integer N such that

$$x_n \in S_\varepsilon(x) \quad \text{for all } n > N.$$

More preciously, a sequence $\{x_n\}$ in a metric space X converges to a point $x \in X$ if the sequence $\{d(x_n, x)\}$ of real numbers converges to 0 as $n \to \infty$.

We use the following symbols to write a convergent sequence.

$$x_n \to x \qquad \text{or} \qquad \lim_{n\to\infty} x_n = x$$

and we express it by saying that x_n approaches x or that x_n converges to x.

Theorem 2.1. *A sequence in a metric space cannot converge to more than one limit point. In other words, in a metric space, every convergent sequence has a unique limit.*

Proof. Let (X, d) be a metric space and $\{x_n\}$ be a convergent sequence in X. Suppose to the contrary that $\{x_n\}$ converges to two distinct points x and y. Then, for each $\varepsilon > 0$, there exist positive integers N_1 and N_2 such that

$$d(x_n, x) < \frac{\varepsilon}{2} \quad \text{for all } n > N_1$$

and

$$d(x_n, y) < \frac{\varepsilon}{2} \quad \text{for all } n > N_2.$$

By triangle inequality, we have

$$\begin{aligned} d(x, y) &\le d(x_n, x) + d(x_n, y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{for all } n > N = \max\{N_1, N_2\}. \end{aligned}$$

It follows that $x = y$. Hence the limit is unique. $\square$

Theorem 2.2. *Let (X, d) be a metric space and A be a subset of X. Then*

(i) A point $x \in X$ is a limit point of A if there exists a sequence $\{x_n\}$ of points of A, none of which equals x, such that $\{x_n\}$ converges to x.
(ii) The set A is closed if and only if every convergent sequence of points of A has its limit in A.

Proof. (i) Let $x \in X$ be a limit point of A. Construct a sequence $\{x_n\}$ by recursion as follows:

Since $x \in X$ is a limit point of A, we have $(S_1(x) - \{x\}) \cap A \neq \emptyset$. So, we can take $x_1 \in (S_1(x) - \{x\}) \cap A$. Likewise the point $x_1, x_2, \ldots, x_n$ have been chosen such that

$$x_i \in \left(S_{1/i}(x) - \{x\}\right) \cap A, \quad \text{for } i = 1, 2, \ldots, n.$$

Still $\left(S_{\frac{1}{n+1}}(x) - \{x\}\right) \cap A \neq \emptyset$, we can always choose $x_{n+1} \in \left(S_{\frac{1}{n+1}}(x) - \{x\}\right) \cap A$. Replace this process infinitely many times. Thus, the sequence $\{x_n\}$ has been constructed by recursion, all the points of which are in A and name of which equals x.

Now, Let $\varepsilon > 0$ be given and let N be a positive integer such that $N > \frac{1}{\varepsilon}$. Then

$$x_n \in S_{\frac{1}{n}}(x) \subset S_\varepsilon(x), \quad \text{for all } n > N.$$

Here $\{x_n\}$ converge to x.

Conversely, assume that there is a sequence $\{x_n\}$ of point of A, none of which equals x, such that $\{x_n\}$ converges to x. Then for every $\varepsilon > 0$, there exists a positive integer N such that

$$x_n \in S_\varepsilon(x), \quad \text{for all } n > N.$$

Therefor $(S_\varepsilon(x) - \{x\}) \cap A \neq \emptyset$ which implies that x is a limit point of A.

(ii) Suppose that A is closed and $\{x_n\}$ is a sequence of points of A which converges to a point x (say) in X. Then we have to show that $x \in A$.

If the range of the sequence $\{x_n\}$ is infinite, then it follows that x is a limit point of this set. Since A is closed, we have $x \in A$.

If, on the other hand, the range of the sequence $\{x_n\}$ is finite, then $x_n = x$ for all $n \geq N$, since $\{x_n\}$ is a convergent sequence. Since each term of the sequence belongs to A, we have $x \in A$.

Conversely, assume that each convergent sequence of points of A converges to a point of A. We shall show A is closed by showing that it contains all its limits points.

Let x be a limit point of A. Then by part (i), there is a sequence $\{x_n\}$ of points of A, none of which equals x, such that $x_n \to x$. By hypothesis $x \in A$. Hence A is closed. □

Problem 2.1. *Show that the limit of a convergent sequence of distinct points in a metric space is a limit of the range of the sequence.*

Proof. Let $\{x_n\}$ be a sequence in a metric space such that $x_n \to x$ and let A be the range of the sequence $\{x_n\}$. Then we have to show that x is a limit point of A.

Suppose that x is not a limit point of A. Then there exists an open sphere $S_\varepsilon(x)$ such that

$$(S_\varepsilon(x) - \{x\}) \cap A = \emptyset,$$

that is, $S_\varepsilon(x)$ contains no point of A other that x. Since x is a limit point of the sequence, for each $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_n, x) < \varepsilon \quad \text{or} \quad x_n \in S_\varepsilon(x), \quad \forall n > N$$

which is a contradiction. Hence the result is proved. □

Definition 2.3. Let (x, d) be a metric space. A sequence $\{x_n\}$ in X is said to be a *Cauchy sequence* if for each $\varepsilon > 0$, there exist a positive integer N such that

$$d(x_n, x_m) < \varepsilon \quad \text{for all } n, m > N.$$

Theorem 2.3. *Every convergent sequence in a metric space is a Cauchy sequence.*

Proof. Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that $x_n \to x$ as $x \to \infty$. Then for each $\varepsilon > 0$, there exist a positive integer N such that

$$d(x_n, x) < \frac{\varepsilon}{2}, \quad \text{for all } n > N.$$

By triangle inequality

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x_n, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } n, m > N. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy Sequence. □

Remark 2.1. Every Cauchy sequence need not be convergent.

Example 2.1. 1. Consider the sequence $\{x_n\}$ is the usual metric space $\mathbb{Q}$, where

$$\begin{aligned} x_1 &= 0.1 \\ x_2 &= 0.101 \\ x_3 &= 0.101001 \\ x_3 &= 0.1010010001 \\ \cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

It is easy to verify that $\{x_n\}$ is a Cauchy sequence which does not converge in $\mathbb{Q}$.

2. Let $X = (0,1]$ be a metric space with the usual metric and $\{x_n\}$, where $x_n = \dfrac{1}{n}$, be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence since for each $\varepsilon > 0$, we have

$$d(x_m, x_n) = \left|\frac{1}{m} - \frac{1}{n}\right| < \varepsilon, \quad \text{for all } m, n > \frac{1}{\varepsilon}.$$

On other hand, $x_n \to 0 \notin X$.

Remark 2.2. In above Example 2, if we take $X = [0,1]$, then the sequence $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ is Cauchy as well convergent.

Theorem 2.4. *Let (X,d) be a metric space and let $\{x_n\}$ be a convergent sequence in X such that $x_n \to x$ as $n \to \infty$. If $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then $x_{n_k} \to x$ as $k \to \infty$.*

Proof. Since every convergent sequence is Cauchy, we have

$$\begin{aligned} d(x_{n_k}, x) &\le d(x_{n_k}, x_n) + d(x_n, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \quad \text{for all } n, n_k > N. \end{aligned}$$

Hence $x_{n_k} \to x$ as $k \to \infty$. □

Remark 2.3. If a subsequence of a sequence in a metric space (X,d) is convergent, then the sequence itself need not be convergent.

Example 2.2. Consider the sequence $\{x_n\}$, where $x_n = (-1)^n$, in $\mathbb{R}$ with used metric. Let $\{x_{2^n}\}$ be a subsequence of the sequence $\{x_n\}$ given by

$$x_{2^n} = 1, \quad \text{for all } n,$$

such that $x_{2^n} \to 1$ as $n \to \infty$. But $\{x_n\}$ is not a convergent sequence.

Theorem 2.5. *Let $\{x_n\}$ be a Cauchy sequence in a metric space (X,d). Then $\{x_n\}$ is convergent if and only if it has a convergent subsequence.*

Proof. Let$\{x_{n_k}\}$ be a convergent subsequence of the sequence $\{x_n\}$. Suppose that $x_{n_k} \to x$ as $k \to \infty$. Then for each $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_{n_k}, x) < \frac{\varepsilon}{2}, \quad \text{for all } n_k > N.$$

Since $\{x_n\}$ is a Cauchy sequence, we have

$$d(x_{n_k}, x_n) < \frac{\varepsilon}{2}, \quad \text{for all } n, n_k > N.$$

By triangle inequality, we have

$$\begin{aligned} d(x_n, x) &\le d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{for all } n > N. \end{aligned}$$

Hence $\{x_n\}$ is convergent.

The converse part follows from Theorem 3.2.4. □

Problem 2.2. *Prove that Cauchy sequence is a discrete metric space is convergent.*

Proof. Let (X, d) be a discrete metric space and let $\{x_n\}$ be a Cauchy sequence in X. Recall that d is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Let $\varepsilon = \dfrac{1}{2}$. There, since $\{x_n\}$ is a Cauchy sequence, there exist a positive integer N such that

$$d(x_n, x_m) < \frac{1}{2}, \quad \text{for all } n, m > N.$$

From the definition of d, we have $x_n = x_m$ for all $n, m > N$. In other words, $\{x_n\}$ is of the form $\{x_1, x_2, \ldots, x_N, x, x, \ldots\}$, that is, constant form some term on. Hence $x_n \to x$ as $n \to \infty$. □

Problem 2.3. *Let (X, d) be a metric space. If $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $x_n \to x$ and $y_n \to y$, then prove that $d(x_n, y_n) \to d(x, y)$.*

Proof. Since $x_n \to x$ and $y_n \to y$, for each $\varepsilon > 0$, there exist positive integers N_1 and N_2 such that

$$d(x_n, x) < \frac{\varepsilon}{2}, \quad \text{for all } n > N_1$$

and

$$d(x_m, x) < \frac{\varepsilon}{2}, \quad \text{for all } m > N_2.$$

Now, if $N = \max\{N_1, N_2\}$, then for all $n, m > N$,

$$\begin{aligned} |d(x_n, y_n) - d(x, y)| &\le |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \\ &\le d(y_n, y) + d(x_n, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $d(x_n, y_n) \to d(x, y)$. □

Problem 2.4. *Let d and d^* be two metrics on the same underlying set X and there exist two real numbers $K_1,\ K_2 > 0$ such that*

$$K_1 d(x,y) \le d^*(x,y) \le K_2 d(x,y), \quad \textit{for all } x, y \in X.$$

Prove that the Cauchy sequence in (X,d) and (X,d^*) are the same.

Problem 2.5. *Let $\{x_n\}$ be a Cauchy sequence in a metric space (X,d) and let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Show that $\lim\limits_{n\to\infty} d(x_n, x_{n_k}) = 0$.*

Proof. Let $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence in X, there exists a positive integer N such that

$$d(x_n, x_m) < \varepsilon, \quad \text{for all } n, m > N - 1.$$

Now $n_N \ge N > N - 1$ and therefore

$$d(x_N, x_{n_N}) < \varepsilon.$$

In other words $\lim\limits_{n\to\infty} d(x_n, x_{n_k}) = 0$. □

Problem 2.6. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space (X,d) such that $\{y_n\}$ is a Cauchy and $d(x_n,y_n) \to 0$ as $n \to \infty$. Then prove that*

(i) $\{x_n\}$ is a Cauchy sequence in X;
(ii) $\{x_n\}$ Converges to, say, $x \in X$ if and only if $\{y_n\}$ Converges to x.

Proof. (i) Let $\varepsilon > 0$. Since $\{y_n\}$ is a Cauchy sequence, there exists a positive integer N_1 such that

$$d(y_m, y_n) < \frac{\varepsilon}{3}, \quad \text{for all } m, n > N_1.$$

By hypothesis, $d(x_n, y_n) \to 0$ as $n \to \infty$, hence there exists a positive integer N_2 such that $\dfrac{1}{N_2} < \dfrac{\varepsilon}{3}$ and

$$d(x_n, y_n) < \frac{\varepsilon}{3}, \quad \text{for all } n > N_2.$$

By triangle inequality, we have

$$d(x_m, x_n) \le d(x_m, y_m) + d(y_m, y_n) + d(y_n, x_n).$$

Hence for all $n, m > N_2$, we have

$$d(x_m, x_n) < \frac{\varepsilon}{3} + d(y_m, y_n) + \frac{\varepsilon}{3}.$$

Let $N_0 = \max\{N_1, N_2\}$. Then for all $n, m > N_0$, we have

$$d(x_m, x_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence.

(ii) By triangle inequality, we have

$$d(y_n, x) \leq d(y_n, x_n) + d(x_n, x)$$

and hence

$$\lim_{n\to\infty} d(y_n, x) \leq \lim_{n\to\infty} d(y_n, x_n) + \lim_{n\to\infty} d(x_n, x).$$

But $\lim_{n\to\infty} d(y_n, x_n) = 0$ and if $\lim_{n\to\infty} d(x_n, x) = 0$, we have $\lim_{n\to\infty} d(y_n, x) = 0 \Rightarrow y_n \to x$ as $n \to \infty$. $\square$

Problem 2.7. *Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a metric space (X, d). Then prove that $\{d(x_n, y_n)\}$ in a convergent sequence.*

Problem 2.8. *Let (X, d) be a metric space and let d^* be the metric on X defined by*

$$d^*(x, y) = \min\{1, d(x, y)\}.$$

Show that $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if it is a Cauchy sequence in (x, d^*).

2.3 Complete Metric Spaces

Definition 2.4. A metric space (x, d) is said to be if every Cauchy sequence in X converges to a point in X.

Remark 2.4. In view of Theorem 3.2.5, a metric space (X, d) is complete if and only if every Cauchy sequence in X has a convergent subsequence.

Example 2.3. 1. The usual metric spaces $\mathbb{R}$ and $\mathbb{C}$ are complete.

2. The set of integer $\mathbb{I}$ with usual metric is a complete metric space.

Let $\{x_n\}$ be a Cauchy sequence of integers, that is, each term of the sequence belongs to $\mathbb{I} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The sequence must be of the form $\{x_1, x_2, x_3, \ldots, x_n, x, x, x, \ldots\}$. For if we choose $\varepsilon = \frac{1}{2}$, then

$$x_n, x_m \in \mathbb{I} \text{ and } |x_n - x_m| < \frac{1}{2} \text{ implies } x_n = x_m$$

Hence the sequence $\{x_1, x_2, \ldots, x_n, x, x, x, \ldots\}$ will converge to x.

3. Let $\mathbb{R}^n$ be an Euclidean space with the metric

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$$

for all $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, is a complete metric space.

Let $\{x_m\}$ be a Cauchy sequence in $\mathbb{R}^n$, where $x_m = \left(\alpha_1^{(m)}, \alpha_2^{(m)}, \ldots, \alpha_n^m\right)$ that is, $x_1 = \left(\alpha_1^{(1)}, \alpha_2^{(2)}, \ldots, \alpha_n^{(1)}\right)$, $x_2 = \left(\alpha_1^{(2)}, \alpha_2^{(2)}, \ldots, \alpha_n^{(2)}\right)$. Then for every $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_m, x_p) = \left[\sum_{i=1}^{n} \left(\alpha_i^{(m)} - \alpha_i^{(p)}\right)^2\right]^{\frac{1}{2}} < \varepsilon, \quad \text{for all } p, m > N \qquad (*)$$

On squaring both the sides, we get

$$\sum_{i=1}^{n} \left(\alpha_i^{(m)} - \alpha_i^{(p)}\right)^2 < \varepsilon^2 \Rightarrow \left(\alpha_i^{(m)} - \alpha_i^{(p)}\right)^2 < \varepsilon^2 \Rightarrow \left|\alpha_i^{(m)} - \alpha_i^{(p)}\right| < \varepsilon$$

for all $m, p > N$, $\quad (i = 1, 2, \ldots, n)$.

It follows that for each fixed i $\;(1 \leq i \leq n)$, the sequence $\{\alpha_i^{(m)}\}_m$ is a Cauchy sequence in the usual metric space $\mathbb{R}$. Since $\mathbb{R}$ Complete, it converges to some point in $\mathbb{R}$. Let $\alpha_i^{(m)} \to \alpha_i$ as $m \to \infty$ for each $i = 1, 2, \ldots, n$, that is,$\alpha_{(1)}^m \to \alpha_1, \alpha_{(2)}^m \to \alpha_2, \alpha_{(n)}^m \to \alpha_n$. Then for each $i = 1, 2, \ldots, n$, $\left|\alpha_i^{(m)} - \alpha_i\right| \to 0$ as $m \to \infty$. Let $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then clearly $x \in \mathbb{R}^n$. Hence

$$d(x_m, x) = \left[\sum_{i=0}^{n} \left(\alpha_i^{(m)} - \alpha_i\right)^2\right]^{\frac{1}{2}} \to 0, \quad \text{as } m \to \infty.$$

Thus $x_m \to x$ as $m \to \infty$ and therefore $\{x_m\}$ is a convergent sequence. Therefore $\mathbb{R}^n$ is complete.

4. The unitary space $\mathbb{C}$ is a complete metric space (Verify).

5. By Exercise 1, every Cauchy sequence is convergent is a discrete metric space and hence every discrete metric space is complete.

Problem 2.9. *Prove or disprove that $\mathbb{R}^n$ is a complete metric space with respect to the following metrics: For all $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^m$;*

(i) $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$;

(ii) $d_\infty(x, y) = \max_{1 \leq i \leq n} \{|x_i - y_i|\}$.

Example 2.4. The space ℓ^p $(1 \leq p < \infty)$ of all sequences $\{\alpha_i\}$ of real or complex numbers such that $\sum\limits_{n=1}^{\infty} |\alpha|^p < \infty$ with the metric

$$d_p(x, y) = \left(\sum_{i=1}^{\infty} |\alpha_i - \beta_i|^p \right)^{\frac{1}{p}}, \quad \text{for all } x, y \in \ell^p$$

is a complete metric space.

Let $\{x_n\}$ be a Cauchy sequence in ℓ^p, where $x_m = \{\alpha_i^{(m)}\}_i$ such that $\sum\limits_{n=1}^{\infty} |\alpha_i^{(m)}|^p < \infty$, $(m = 1, 2, \ldots)$. Then for each $\varepsilon > 0$, there exist a positive integer N such that

$$d_p(x_m, x_n) = \left(\sum_{i=1}^{\infty} \left| \alpha_i^{(m)} - \alpha_i^{(n)} \right|^p \right)^{\frac{1}{p}} < \varepsilon, \quad \text{for all } m, n > N \qquad (*)$$

and thus

$$\left| \alpha_i^{(m)} - \alpha_i^{(n)} \right| < \varepsilon, \quad \text{for all } m, n > N, \quad (i = 1, 2, \ldots).$$

This shows that for each fixed i $(1 \leq i < \infty)$ the sequence $\{\alpha_i^{(m)}\}_m$ is a Cauchy sequence in $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Since $\mathbb{K}$ is complete, it converges in $\mathbb{K}$. Let $\alpha_i^{(m)} \to \alpha_i$ as $m \to \infty$. Using these limits, we define $x = (\alpha_1, \alpha_2, \ldots)$ and show that $x \in \ell^p$ and and $x_m \to x$.

From (*), we get

$$\sum_{i=1}^{k} \left| \alpha_i^{(m)} - \alpha_i^{(n)} \right|^p < \varepsilon^p, \quad \text{for all } m, n > N \quad (k = 1, 2, \ldots).$$

Letting $n \to \infty$, we obtain

$$\sum_{i=1}^{k} \left| \alpha_i^{(m)} - \alpha_i \right|^p < \varepsilon^p, \quad \text{for all } m > N \quad (k = 1, 2, \ldots)$$

which, on letting $k \to \infty$, gives

$$\sum_{i=1}^{k} \left| \alpha_i^{(m)} - \alpha_i \right|^p < \varepsilon^p, \quad \text{for all } m > N. \qquad (**)$$

This shows that

$$x_m - x = \left\{ \alpha_i^{(m)} - \alpha_i \right\} \in \ell^p.$$

Since $x_m \in \ell^p$, it follows by means of Minkowski inequality[1]

$$x = x_m + (x - x_m) \in \ell^p.$$

Thus $x \in \ell^p$. Furthermore, from (**), we obtain

$$d_p(x_m) < \varepsilon, \quad \text{for all } m > N$$

which verifies that $x_m \to x$ in ℓ^p. Hence $\ell^p \ (1 \leq p < \infty)$ is a complete metric space.

Example 2.5. The sequence space $\ell^\infty = \left\{ \{\alpha_i\} \subseteq \mathbb{K} : \sup\limits_{1 \leq i < \infty} |\alpha_i| < \infty \right\}$, with the metric $d(x, y) = \sup\limits_{1 \leq i < \infty} |\alpha_i - \beta_i|$, where $x = \{\alpha_i\}$, $y = \{\beta_i\}$, is a complete metric space.

Let $\{x_m\}$ be a Cauchy sequence in ℓ^∞, where $x_m = \{\alpha_i^{(m)}\}_i$ such that $\sup\limits_{1 \leq i < \infty} \left|\alpha_i^{(m)}\right|$
$< \infty$, $(m = 1, 2, \ldots)$. Then for each $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_m, x_n) = \sup_{1 \leq i < \infty} \left|\alpha_i^{(m)} - \alpha_i^{(n)}\right| < \varepsilon, \quad \text{for all } m, n > N$$

and thus

$$\left|\alpha_i^{(m)} - \alpha_i^{(n)}\right| < \varepsilon, \quad \text{for all } m, n > N, \quad (i = 1, 2, \ldots). \tag{$*$}$$

This shows that for each fixed $i \ (1 \leq i < \infty)$, the sequence $\{\alpha_i^{(m)}\}_m$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is complete, it converges in $\mathbb{K}$. Let $\alpha_i^{(m)} \to \alpha_i$ as $m \to \infty$. Using these limits, we define $x = (\alpha_1, \alpha_2, \ldots)$ and show that $x \in \ell^\infty$ and $x_m \to x$.

Letting $n \to \infty$ in (*), we get

$$\left|\alpha_i^{(m)} - \alpha_i\right| < \varepsilon, \quad \text{for all } m > N \quad (i = 1, 2, \ldots). \tag{$**$}$$

Since $x_m = \{\alpha_i^{(m)}\}_i \in \ell^\infty$, there is a real number k_m such that $\left|\alpha_i^{(m)}\right| \leq k_m$ for all i. Therefore,

$$\begin{aligned} |\alpha_i| &= \left|\alpha_i - \alpha_i^{(m)} + \alpha_i^{(m)}\right| \\ &\leq \left|\alpha_i^{(m)} - \alpha_i\right| + |\alpha_i^m| \\ &\leq \varepsilon + k_m, \quad \text{for all } m > N \quad (i = 1, 2, \ldots). \end{aligned}$$

[1] Minkowski's Inequality: Let $1 \leq p < \infty$. If $(x_1, \ldots), (y_1, \ldots) \in \ell^p$, that is, $\sum\limits_{i=1}^{\infty} |x_i|^p < \infty$ and $\sum\limits_{i=1}^{\infty} |y_i|^p < \infty$, then

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{\frac{1}{p}}.$$

This inequality being true for each i and the right hand side being independent of i, it follows that $\{\alpha_i\}$ is a bounded sequence of numbers. This implies that $x = \{\alpha_i\} \in \ell^\infty$.

Furthermore, from (**), we obtain

$$d(x_m, x) = \sup_{1 \le i < \infty} \left| \alpha_i^{(m)} - \alpha_i \right| < \varepsilon, \quad \text{for all } m > N.$$

This Shows that $x_m \to x$ in ℓ^∞. Hence ℓ^∞ is a complete metric space.

Example 2.6. The space $C[a, b]$ of all continuous real valued functions defined on $[a, b]$ with the metric

$$d_\infty(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|$$

is a complete metric space.

Let $\{f_m\}$ be a Cauchy sequence in $C[a, b]$. Then for each $\varepsilon > 0$, there exists a positive integer N such that

$$d_\infty(f_m, f_n) = \max_{t \in [a,b]} |f_m(t) - f_n(t)| < \varepsilon, \quad \text{for all } m, n > N \qquad (*)$$

Therefore, for any fixed $t_0 \in [a, b]$, we get

$$|f_m(t_0) - f_n(t_0)| < \varepsilon, \quad \text{for all } m, n > N.$$

This shows that $\{f_m(t_0)\}$ is a Cauchy sequence in $\mathbb{R}$. But since $\mathbb{R}$ is complete, this sequence converges. Let $f_m(t_0) \to f(t_0)$ as $m \to \infty$. In this way, we can associate to each $t \in [a, b]$ a unique real number $f(t)$. This defines (pointwise) a function f on $[a, b]$. Now, we show that $f \in C[a, b]$ and $f_m \to f$.

From (*), we have

$$|f_m(t) - f_n(t)| < \varepsilon, \quad \text{for all } m, n > N \text{ and for all } t \in [a, b].$$

Letting $n \to \infty$, we get

$$|f_m(t) - f(t)| < \varepsilon, \quad \text{for all } m > N \text{ and for all } t \in [a, b]. \qquad (**)$$

This verifies that the sequence $\{f_m\}$ of continuous functions converges uniformly to the function f on $[a, b]$ and the hence limit function f is a continuous function on $[a, b]$. As such $f \in C[a, b]$.

Also, from (**), we have

$$\max_{t \in [a,b]} |f_m(t) - f(t)| < \varepsilon, \quad \text{for all } m > N.$$

Thus $d_\infty(f_m, f) < \varepsilon$, for all $m > N$ and therefore $f_m \to f$ as $m \to \infty$. Hence $C[a, b]$ is a complete metric space.

But the space $C[a,b]$, with $a = 0$ and $b = 1$ is not a complete metric space with respect to the metric

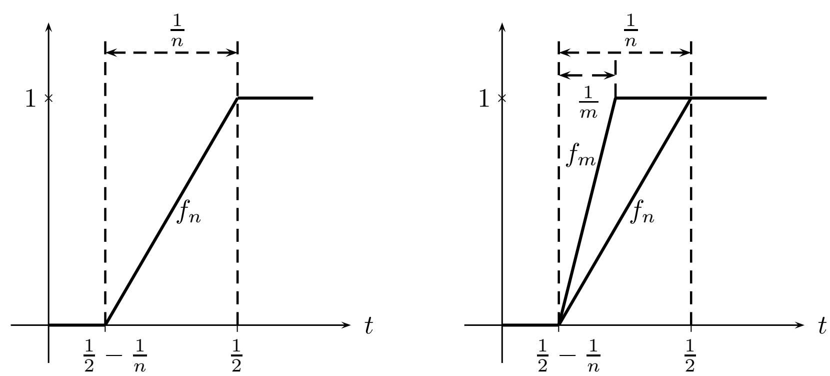
$$d_1(f,g) = \int_0^1 |f(t) - g(t)|dt.$$

Let $\{f_n\}$ be a sequence in $C[0,1]$, where

$$f_n(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{2} - \frac{1}{2} \\ nt - \frac{1}{2}n + 1, & \frac{1}{2} - \frac{1}{n} < t \le \frac{1}{2} \\ 1, & \frac{1}{2} < t \le 1. \end{cases}$$

We shall know that $\{f_n\}$ is a Cauchy sequence but does not converge in $(C[0,1], d_1)$.

Note that $d_1(f_n, f_m) < \dfrac{1}{n} + \dfrac{1}{m} < \varepsilon$, for all $n, m > N$, where N is a positive integer such that $N > \frac{2}{\varepsilon}$. This shows that $\{f_n\}$ is a Cauchy sequence.



$d_1(f_n, f_m)$ represents the area of the triangle

Let, if possible, $f \in C[0,1]$ be such that $d_1(f_n, f) \to 0$. But

$$d_1(f_n, f) = \int_0^{\frac{1}{2}-\frac{1}{n}} |f(t)|\, dt + \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} |f_n(t) - f(t)|\, dt + \int_{\frac{1}{2}}^1 |1 - f(t)|\, dt. \tag{$***$}$$

Since the integrands are non-negative, so is the each integral on the right hand side of (***). Consequently, we have

$$\left.\begin{aligned} \lim_{n\to\infty} \int_0^{\frac{1}{2}-\frac{1}{n}} |f(t)|dt = 0 \\ \int_{\frac{1}{2}}^1 |1 - f(t)|dt = 0 \end{aligned}\right\} \Rightarrow f(t) = \begin{cases} 0, & \text{if } 0 \le t < \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Therefore f is not continuous on $[0,1]$, that is, $f \notin C[0,1]$. Hence $C[0,1]$ is not a complete metric space.

Example 2.7. The space $\mathbb{Q}$ with usual metric of absolute value is not complete.

Example 2.8. The metric space (X,d), where $X = (0,1]$ and d is the usual metric on X, is not complete.

Problem 2.10. *Prove that* $[0,1)$ *as a subspace of the discrete metric space* $\mathbb{R}$ *is complete.*

Theorem 2.6. *Let* (Y, d_Y) *be a subspace of a metric space* (X,d). *If* Y *is complete, then it is closed.*

Proof. Suppose that Y is a complete subspace. To prove Y is closed, it is sufficient to show that Y contains all its limit points.

Let x be a limit point of Y. Then by Theorem 3.2.3 "Let (x,d) be a metric space and $A \subseteq X$. A point $x \in X$ is a limit point of A if and only if there is a sequence of distinct points of A which converges to x", there exist a sequence $\{x_n\}$ of distinct points of A which converges to x. Since each convergent sequence is Cauchy, it is a Cauchy sequence. Also since A is complete, the limit of this sequence, say x, must lie in A. Thus A is closed. □

Theorem 2.7. *Let* (X,d) *be a complete metric space and* (Y, d_y) *a subspace of* (x,d). *Then* Y *is complete if and only if it is closed.*

Proof. If Y is a complete subspace of (X,d), then by theorem 3.3.1, it is closed.

Conversely, assume that Y is a closed subspace of a complete metric space X. Let $\{x_n\}$ be a Cauchy sequence of points of Y. Since X is complete, this sequence converges to a point x belonging to X. By Theorem 3.2.2 (ii) "$A \subseteq X$ is closed if and only if each convergent sequence of points of A converges to a point of A", and since A is closed, $x \in A$. Thus each Cauchy sequence of point of A converges to a points of A. Hence A is complete. □

Theorem 2.8 (Cantor's Intersection Theorem). *Let* (X,d) *be a complete metric space and let* $\{F_n\}$ *be a decreasing sequence (that is,* $F_{n+1} \subseteq F_n$*) of nonempty closed subsets of* X *such that* $\delta(F_n) \to 0$ *as* $n \to \infty$. *Then the intersection* $\bigcap_{x=1}^{\infty} F_n$ *contains exactly one point.*

Proof. Construct a sequence $\{x_n\}$ in X by selecting a point $x_n \in F_n$ for each n. Since the sets F_n are nested, that is, $F_{n+1} \subseteq F_n$, we have $x_n \in F_n \subseteq F_m$, for all $n > m$. We claim that $\{x_n\}$ is a Cauchy sequence.

Let $\varepsilon > 0$ be given. Since $\delta(F_n) \to 0$, there exists a positive integer N such that $\delta(F_n) < \varepsilon$. Since $\{f_n\}$ is a decreasing sequence, we have $F_m,\ F_n \subseteq F_N$ for all $m, n \geq N$. Therefore, $x_n, x_m \in F_N$ for all $n, m \geq N$ and thus, we have

$$d(x_n, x_m) \leq \delta(F_n) < \varepsilon, \quad \text{for all } n, m \geq N.$$

Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $x \in X$ such that $x_n \to x$. We claim that $x \in \bigcap\limits_{n=1}^{\infty} F_n$.

Let n be fixed. Then the subsequence $\{x_n, x_{n+1}, \ldots\}$ of $\{x_n\}$ is contained is F_n and still converges to x, since every subsequence of a convergent sequence is convergent. But F_n being a closed subspace of the complete metric space (X, d), it is complete and so $x \in F_n$. This is true for each $n \in \mathbb{N}$. Hence $x \in \bigcap\limits_{n=1}^{\infty} F_n$, that is, $\bigcap\limits_{n=1}^{\infty} F_n \neq \emptyset$.

Finally, to establish that x is the only point in the intersection $\bigcap\limits_{n=1}^{\infty} F_n$, let $y \in \bigcap\limits_{n=1}^{\infty} F_n$. Then x and y both are in F_n for each n. Therefore,

$$0 \leq d(x, y) \leq \delta(F_n) \to 0, \quad \text{as } n \to \infty.$$

Thus $d(x, y) = 0$ and hence $x = y$. □

Remark 2.5. The assertion in Theorem 3.3.3 may not be true if either of the conditions

(a) each F_n is closed
(b) $\delta(F_n) \to 0$ as $n \to \infty$

is dropped.

Example 2.9. Consider the usual metric space $\mathbb{R}$ which, of course, is complete.

(a) Take $F_n = [n, \infty)$. Note that $\{F_n\}$ is a sequence of nonempty closed sets such that $d(F_n) \not\to 0$ as $n \to \infty$ and that $\bigcap\limits_{n=1}^{\infty} F_n = \emptyset$.
(b) Take $F_n = (0, \frac{1}{n}]$. Note that $\{F_n\}$ is a decreasing sequence (that is, $F_{n+1} \subseteq F_n$) of nonempty set which are not closed, $\delta(F_n) \to 0$ as $n \to \infty$ and $\bigcap\limits_{x=1}^{\infty} F_n = \emptyset$.

Now, we have the converse of Theorem 3.3.3.

Theorem 2.9. *If in a metric space (X, d) every decreasing sequence $\{F_n\}$ of nonempty closed sets with $\delta(F_n) \to 0$ as $n \to 0$ has exactly on one point in its intersection, then (X, d) is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Let $G_1 = \{x_1, x_2, \ldots\}$, $G_2 = \{x_2, x_3, \ldots\}, \cdots, G_n = \{x_n, x_{n+1}, \ldots\}$.

Since $\{x_n\}$ is a Cauchy sequence, for a given $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_m, x_n) < \varepsilon, \quad \text{for all } m, n \geq N.$$

But $m, n \geq N$, we have $x_m, x_n \in G_n$ and therefore $d(x_m, x_n) < \varepsilon$, which implies that $\delta(G_n) < \varepsilon$. For $\neq N$, we have $G_n \subseteq G_N$ and thus $\delta(G_n) \leq \delta(G_N) < \varepsilon$. Therefore, $\delta(G_n) \to 0$ as $n \to \infty$.

Since $\delta(G_n) = \delta(\overline{G_n})$, we have $\delta(\overline{G_n}) \to 0$ as $n \to \infty$. Taking $F_n = \overline{G_n}$, then $\{F_n\}$ is a decreasing sequence of nonempty closed sets with $\delta(F_n) \to 0$ as $n \to \infty$. Then by hypothesis, there exist an $x \in X$ such that $x \in \bigcap\limits_{n=1}^{\infty}$. Therefore, $d(x, x_n) \leq \delta(F_n)$ for all n and so $d(x, x_n) \leq \delta(F_n) \to 0$ as $n \to \infty$. Hence $x_n \to x$ in X. Thus, (X, d) is complete. $\square$

Chapter 3

Separable Spaces

3.1 Countability

Definition 3.1. Let (X, d) be a metric space and $\mathcal{O}$ be the family of all open subset of X. A subfamily $\mathcal{B}$ of open subsets of X, that $\mathcal{B} \subseteq \mathcal{O}$, is said to be a *base* or *basis* for $\mathcal{O}$ if every open set $G \in \mathcal{O}$ is the union of members of $\mathcal{B}$.

Before giving the examples of a base for a family of open sets, we mention the following characterization of a base.

Theorem 3.1. *Let (X, d) be a metric space and let $\mathcal{O}$ be the family of all open subset of X. A subclass $\mathcal{B}$ of $\mathcal{O}$, that is, $\mathcal{B} \subseteq \mathcal{O}$, is a base for $\mathcal{O}$ if and only if for any point x belonging to an open set G, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq G$.*

Proof. Let $\mathcal{B}$ be a base for $\mathcal{O}$ and G be any open set in X, that is, $G \in \mathcal{O}$. Then by definition, G is the union of member of $\mathcal{B}$. Let $x \in G$. Since G in the union of members of $\mathcal{B}$, there exists a set B_x in $\mathcal{B}$ such that $x \in B_x \subseteq G$.

Conversely, let G be any arbitrary open set. Then by hypothesis, for any point $x \in G$, there exists B_x in $\mathcal{B}$ such that $x \in B_x \subseteq G$.

Clearly, $G = \bigcup\{B_x : x \in G \text{ and each } B_x \in \mathcal{B}\}$. Then every open set is the union of members of $\mathcal{B}$. □

Example 3.1. 1. Let (X, d) be a discrete metric space. Then the collection $\mathcal{B} = \{\{x\} : x \in X\}$ forms a base, since every subset of a discrete metric space is open.

2. The collection of all open intervals forms a base for the family of all open sets in the usual metric space $\mathbb{R}$.

3. The collection of all open spheres forms a base for the family of all open sets in a metric space (X,d).

Definition 3.2. A metric space (X,d) is said to be a *first countable space* (or *first axiom space*) if for every point $x \in X$, there exists a countable family $\{B_n(x)\}$ of open sets containing x such that every open set G containing x also contains a member of $\{B_n(x)\}$, that is, $B_n(x) \subseteq G$ for some n.

Example 3.2. The usual metric space $\mathbb{R}$ is a first countable space. Indeed, we may take $B_n(x) = \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Theorem 3.2. *Every metric space (X,d) is a first countable space.*

Proof. Let $x \in X$ and $x \in \mathbb{N}$. Let $B_n(x) = S_{\frac{1}{n}}(x)$. Then $\{B_n(x)\} = \left\{B_1(x), B_{\frac{1}{2}}(x), B_{\frac{1}{3}}(x), \ldots\right\}$ be a countable collection of open subsets of X each of which contains x. Let G be an open set containing x. Then there exists $B_\varepsilon(x)$ such that $B_\varepsilon(x) \subseteq G$ for same $\varepsilon > 0$. In this case, $B_n(x) \subseteq G$ for each $n > \frac{1}{\varepsilon}$ and hence X in a first countable space. $\square$

Definition 3.3. A metric space (X,d) is said to be a *second countable space* (a *second axiom space*) if there exists a countable base for the family of all open subset of X.

Example 3.3. The usual metric space $\mathbb{R}$ is a second countable space. In fact, the collection of all open intervals (a,b) with a and b as rational point forms a base for the family of all open subset of $\mathbb{R}$.

Remark 3.1. Every second constable metric space is first countable but converse in not true.

Example 3.4. Let (X,d) be a discrete metric space with X is an uncountable set. Then, X is a first countable set not second countable.

3.2 Dense sets and Nowhere Dense sets

Definition 3.4. Let (X,d) be a metric space and A a subset of X. Then A is said to be

(i) *dense* (or *everywhere dense*) in X if $\overline{A} = X$;
(ii) nowhere dense in X if $(\overline{A})^\circ = \emptyset$.

Example 3.5. (i) The set of rational numbers $\mathbb{Q}$ is dense in the usual metric space $\mathbb{R}$, since $\overline{\mathbb{Q}} = \mathbb{R}$.

(ii) In the usual metric space $\mathbb{R}$,

(a) any singleton set,

(b) any finite set,

(c) The sets $\mathbb{N}$ and $\mathbb{Z}$ ($\overline{\mathbb{N}} = \mathbb{N} \cup \mathbb{N}' = \mathbb{N} \cup \emptyset = \mathbb{N}$, $\mathbb{N}^\circ = \emptyset$) are nowhere dense in $\mathbb{R}$.

Theorem 3.3. *Let (X, d) be a metric space and A a subset of X. Then A is nowhere dense in X if and only if $X - \overline{A}$ is dense in X.*

Proof. Since $\overline{X - A} = X - A^\circ$, we have $A^\circ = X - \overline{X - A}$. Replacing A by $\overline{A}$, we get $(\overline{A})^\circ = X - \overline{(X - \overline{A})}$. Therefore, $(\overline{A})^\circ = \emptyset$ if and only if $X = \overline{(X - \overline{A})}$. □

Corollary 3.1. *Let (X, d) be a metric space and A a closed subset of X. Then A is nowhere dense in X if and only if $X - A$ is dense in X.*

Problem 3.1. *Let (X, d) be a matrix space and A a subset of X. Then prove that the following statements are equivalent:*

(i) A is dense in X.
(ii) The only closed superset of A is X.
(iii) The only open set disjoint from A is $\emptyset$.
(iv) A intersects every nonempty open set.
(v) A intersects every open sphere.

Problem 3.2. *Let (X, d) be a metric space and A a subset of X. Then prove that the following statements are equivalent*

(i) A is nowhere dense in X.
(ii) $\overline{A}$ does not contain any nonempty open set.
(iii) Every nonempty open set has a nonempty open subset disjoint from $\overline{A}$.
(iv) Every nonempty open set contains a nonempty open subset disjoint from A.
(v) Every nonempty open set contains an open sphere disjoint from A.

Problem 3.3. *Let A be a metric space and A an open subset of X. Then prove that A is dense in x if and only if $X - A$ is nowhere dense in X.*

Problem 3.4. *Prove that a finite union of nowhere dense sets in a metric space is a nowhere dense set.*

Problem 3.5. *Give an example to show that a countable (infinite) union of nowhere dense sets in a metric space* (X, d) *need not be a nowhere dense set in* (X, d)*.*

Chapter 4

Continuous Functions

4.1 Definition and Characterizations

Definition 4.1. Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is said to be *continuous at a point* $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d(x, x_0) < \delta \quad \text{implies} \quad \rho\left(f(x), f(x_0)\right) < \varepsilon,$$

that is,

$$x \in S_\delta(x_0) \quad \text{implies} \quad f(x) \in S_\varepsilon\left(f(x_0)\right).$$

In other words, f is continuous at a point $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$f\left(S_\delta(x_0)\right) \subseteq S_\varepsilon\left(f(x_0)\right).$$

The function f is said to be *continuous on* X if it is continuous at every point of X.

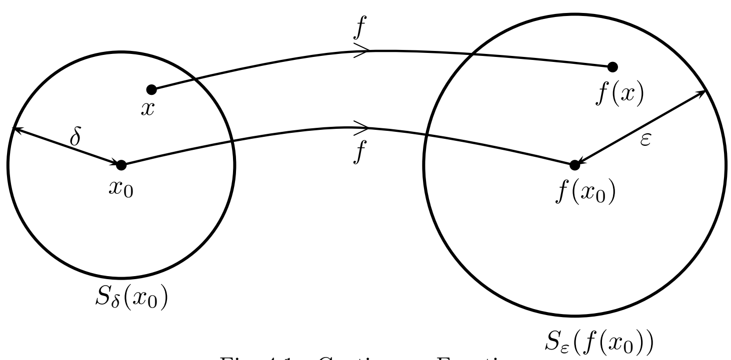


Fig. 4.1 Continuous Function

Example 4.1.

(i) Let (X, d) be a metric space. Then the identity function $I : X \to X$ is continuous on X.
(ii) Let $\mathbb{R}$ be the set of all real numbers with the usual metric, then every constant function is continuous.
(iii) Let (X, d) be a discrete metric space. Then every function $f : X \to Y$ from X to a metric space Y is continuous on X.

Theorem 4.1. *Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous at a point $x_0 \in X$ if and only if for every sequence $\{x_n\} \subset X$, we have $x_n \to x_0$ implies that $f(x_n) \to f(x_n)$.*

Proof. Let f be continuous at a point $x_0 \in X$. Then for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d(x, x_0) < \delta \quad \text{implies} \quad \rho\left(f(x), f(x_0)\right) < \varepsilon.$$

Let $\{x_n\} \subset X$ be a sequence in X such that $x_n \to x_0$. Then there exists a positive integer N such that

$$d(x_n, x_0) < \delta \quad \text{for all} \quad n > N.$$

Hence for all $n > N$, we have

$$\rho\left(f(x_n), f(x_0)\right) < \varepsilon,$$

and therefore $f(x_n) \to f(x_0)$.

Conversely, assume that for every sequence $\{x_n\}$ in X such that $x_n \to x_0$, we have $f(x_n) \to f(x_0)$. Suppose that f is not continuous at x_0. Then there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there is an $x \neq x_0$ satisfying

$$d(x, x_0) < \delta \quad \text{but} \quad \rho\left(f(x), f(x_0)\right) \geq \varepsilon.$$

In particular, for $\delta = \frac{1}{n}$ there is an x_n satisfying

$$d(x_n, x_0) < \frac{1}{n} \quad \text{but} \quad \rho\left(f(x_n), f(x_0)\right) \geq \varepsilon.$$

Then clearly $x_n \to x_0$ but $\{f(x_n)\}$ does not converge to $f(x_0)$. This contradicts to our hypothesis that $f(x_n) \to f(x_0)$. Hence f is continuous at x_0. $\square$

Theorem 4.2. *Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous on X if and only if for each $x \in X$ and for every sequence $\{x_n\} \in X$, we have $x_n \to x$ implies $f(x_n) \to f(x)$.*

Theorem 4.3. *Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous on X if and only if $f^{-1}(G)$ open in X whenever G is open in Y.*

Proof. Assume that f is continuous on X and G is an open set in Y. Then we shall prove that $f^{-1}(G)$ is open in X. If $f^{-1}(G) = \emptyset$, then the result is proved. So, we assume that $f^{-1}(G) \neq \emptyset$. Let $x \in f^{-1}(G)$, then $f(x) \in G$. Since G is open, $S_\varepsilon(f(x)) \subseteq G$ for some $\varepsilon > 0$. By the continuity of f, there exists a $\delta > 0$ such that

$$f\left(S_\delta(x)\right) \subseteq S_\varepsilon(f(x)),$$

and since $S_\varepsilon(f(x)) \subseteq G$, it follows that $f\left(S_\delta(x)\right) \subseteq G$ and therefor $S_\delta(x) \subseteq f^{-1}(G)$. Hence $f^{-1}(G)$ is open.

Conversely, assume that $f^{-1}(G)$ is open in X wherever G is open in Y. Let $x \in X$ be arbitrary and $\varepsilon > 0$ be given. Then $f(x) \in Y$ and $S_\varepsilon\left(f(x)\right) (= G,\text{ say})$ is a open set. Therefore, by assumption $f^{-1}\left(S_\varepsilon(f(x))\right)$ is open and $x \in f^{-1}\left(S_\varepsilon(f(x))\right)$. Consequently, there exists a $\delta > 0$ such that $S_\delta(x) \subseteq f^{-1}\left(S_\varepsilon(f(x))\right)$ and thus $f\left(S_\delta(x)\right) \subseteq S_\varepsilon(f(x))$. Hence f is continuous at x. Since $x \in X$ was an arbitrary, f is continuous on X. □

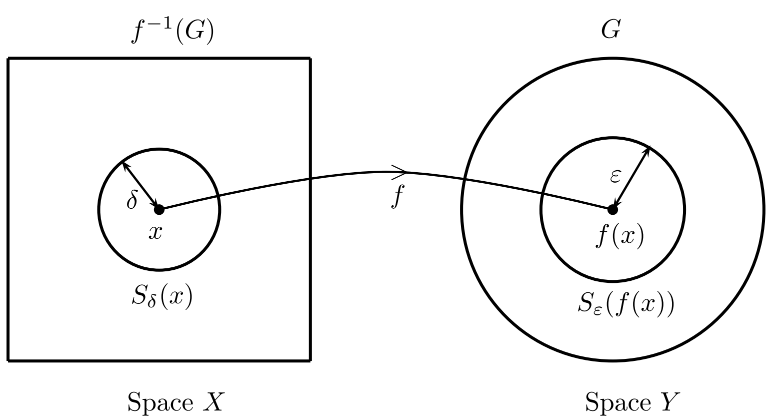


Fig. 4.2

Theorem 4.4. *Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous on X if and only if $f^{-1}(F)$ is closed in X whenever F is closed in Y.*

Proof. Let f be a continuous function and F be a closed in Y. Then $Y \setminus F$ is open in Y and therefore $f^{-1}(Y \setminus F)$ is open in X. Since

$$f^{-1}(F) = X \setminus f^{-1}(Y \setminus F)$$

and $f^{-1}(Y \setminus F)$ is open, it follows that $f^{-1}(F)$ is closed.

Conversely, assume that $f^{-1}(F)$ is closed in X whenever F is closed in Y. Then we shall show that f is continuous. Let G be an open subset of Y. Then $Y \setminus G$ is closed in Y and by hypothesis $f^{-1}(Y \setminus G)$ is closed in X. Since

$$f^{-1}(G) = X \setminus f^{-1}(Y \setminus G)$$

and $f^{-1}(Y \setminus G)$ is closed , we have $f^{-1}(G)$ is open. Hence f is continuous on X. $\square$

Remark 4.1. If f is a continuous function from a metric space (X, d) to another metric space (Y, ρ). Then the image $f(G)$ of an open set G in X need not be open in Y and the image $f(F)$ of a closed set F in X need not be closed in Y.

For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$. Thus, of course, f is continuous on $\mathbb{R}$. Let $G = (-1, 1)$ be an open set in $\mathbb{R}$ but $f(G) = [0, 1)$ is not open in $\mathbb{R}$.

Consider another function $f : [1, +\infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Then, f is continuous on $[1, +\infty)$. Let $A = [1, +\infty)$, then A is a closed subset of $\mathbb{R} = (-\infty, +\infty)$ but $f(A) = (0, 1]$ is not closed in $\mathbb{R}$.

Theorem 4.5. *Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous if and only if $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for every subset A of X.*

Proof. Let f be a continuous function. Then $f^{-1}\left(\overline{f(A)}\right)$ is closed in X, since $\overline{f(A)}$ is closed in Y. Now we have

$$f(A) \subseteq \overline{f(A)} \quad \Rightarrow \quad A \subseteq f^{-1}\left(\overline{f(A)}\right) \quad \Rightarrow \quad \overline{A} \subseteq \overline{f^{-1}\left(\overline{f(A)}\right)}$$

and thus $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$ because $f^{-1}\left(\overline{f(A)}\right)$ is closed. Hence $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.

Conversely, let $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for every subset A of X. We shall prove that f is continuous. Let F be any closed set in Y. Then $\overline{F} = F$. Now, we have

$$f\left(\overline{f^{-1}(F)}\right) \subseteq \overline{f\left(f^{-1}(F)\right)} = \overline{F} = F.$$

Thus implies that $\overline{f^{-1}(F)} \subseteq f^{-1}(F)$. But $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$, therefore $\overline{f^{-1}(F)} = f^{-1}(F)$. Hence $f^{-1}(F)$ is closed in X. Thus f is continuous on X. $\square$

Theorem 4.6. *Let (X, d) and (Y, δ) be metric spaces and $f : X \to Y$ be a function. The following statements are equivalent:*

(a) f is continuous on X.
(b) For each $x \in X$ and for every sequence $\{x_n\}$ in X such that $x_n \to x \Rightarrow f(x_n) \to f(x)$.
(c) $f^{-1}(G)$ is open in X wherever G is open in Y.
(d) $f^{-1}(F)$ is closed in X wherever F is closed in Y.
(e) $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.

Problem 4.1. *Prove that the function $f : (X, d) \to (Y, \rho)$ is continuous on if and only if for every subset B of Y, $\overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{B}\right)$.*

Proof. Let f be continuous and let $A = f^{-1}(B)$. Since $f\left(\overline{A}\right) \subseteq f\left(\overline{A}\right)$ (by Theorem 1.5), we have

$$f(A) \subseteq B \quad \Rightarrow \quad \overline{f(A)} \subseteq \overline{B} \quad \Rightarrow \quad f\left(\overline{A}\right) \subseteq \overline{B}$$

Therefore,

$$\overline{A} \subseteq f^{-1}(\overline{B}) \Rightarrow \overline{f^{-1}(B)} = f^{-1}(\overline{B}).$$

Conversely, let $\overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{B}\right)$ for every subset B of Y. Let F be a closed set in Y. Then $\overline{F} = F$ and by hypothesis, we have

$$\overline{f^{-1}(F)} \subseteq f^{-1}\left(\overline{F}\right) = f^{-1}(F) \quad \text{because } \overline{F} = F.$$

But $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$ and therefore $f^{-1}(F) = \overline{f^{-1}(F)}$. Thus $f^{-1}(F)$ is closed and hence f is continuous on X. $\square$

Problem 4.2. *Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a function. Prove that f is continuous as on X if and only if for every subset B of Y, $f^{-1}(B^\circ) \subseteq \left(f^{-1}(B)\right)^\circ$.*

Proof. Suppose that f is a continuous function. Let B be any arbitrary subset of Y. Then B° is open in Y and by continuity of f, $f^{-1}(B^\circ)$ is open in X. Therefore, $\left(f^{-1}(B)\right)^\circ = f^{-1}(B^\circ)$. But $B^\circ \subseteq B \Rightarrow f^{-1}(B^\circ) \subseteq f^{-1}(B)$ and therefore $\left[f^{-1}(B^\circ)\right]^\circ \subseteq \left[f^{-1}(B)\right]^\circ$. This implies that $f^{-1}(B^\circ) \subseteq \left[f^{-1}(B)\right]^\circ$.

Conversely, let G be an open subset of Y. Then $G^\circ = G$. By the hypothesis

$$\left[f^{-1}(G)\right]^\circ \supseteq f^{-1}(G^\circ) = f^{-1}(G) = f^{-1}(G).$$

But $\left[f^{-1}(G)\right]^\circ \subseteq f^{-1}(G) = f^{-1}(G)$, therefore $f^{-1}(G) = \left[f^{-1}(G)\right]^\circ$ and thus $f^{-1}(G)$ is open in X. Hence f is continuous on X. □

Theorem 4.7. *Let (X, d), $(Y,$
$rho)$ and (Z, σ) be metric spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Then $g \circ f$ is continuous on X.*

Proof. We know that $g \circ f : X \to Z$ Let G be an open set in Z. Then

$$g^{-1}(G) \text{ is open in } Y \quad \Rightarrow \quad f^{-1}\left(g^{-1}(G)\right) \text{ is open in } X$$

$$\Rightarrow \quad \left(f^{-1} \circ g^{-1}\right)(G) \text{ is open in } X \quad \Rightarrow \quad (g \circ f)^{-1}(G) \text{ is open in } X$$

$$\Rightarrow \quad g \circ f \text{ is continuous.}$$ □

4.2 Continuous Functions and Compact Spaces

Theorem 4.8. *Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a continuous function. If A is a compact subset of X, then $f(A)$ is compact in Y.*

Proof. Let $\mathscr{F} = \{G_i\}_i \in \Lambda$ be an open cover of $f(A)$. Then by Theorem 1.3, $f^{-1}(G_i)$ is open in X for each $i \in \Lambda$. Hence $\{A \cap f^{-1}(G_i)\}_{i \in \Lambda}$ from an open cover of A. Since A is compact, there exists a finite set $J = \{1, 2, \ldots, n\}$ of Λ such that

$$A = \bigcup_{k=1}^{n} \left(A \cap f^{-1}(G_k)\right) = A \cap \left(\bigcup_{k=1}^{n} f^{-1}(G_x)\right) = A \cap f^{-1}\left(\bigcup_{k=1}^{n} G_k\right).$$

Therefore, it follows that

$$f(A) \subseteq \bigcup_{k=1}^{n} G_k$$

and hence $\{G_1, G_2, \ldots, G_n\}$ is a finite subcover of $\mathscr{F}$. Thus, $f(A)$ is compact. □

Corollary 4.1. *Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a continuous function. If X is compact, then $f(X)$ is bounded.*

Proof. By above theorem $f(X)$ is compact. Since every compact space is sequentially compact and every sequentially compact space is totally bounded, we have $f(X)$ is totally bounded and hence it is bounded. □

Theorem 4.9. *Let (X, d) and (X, ρ) be metric spaces and $f : X \to Y$ be a continuous function. If X is compact, then $f(F)$ is closed in Y whenever F is closed in X.*

Proof. Let F be a closed subset of X. Since every closed subset of a compact set is compact, by Theorem 2.1, we have $f(F)$ is compact and hence it is closed. □

Theorem 4.10. *Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a continuous function. If f is bijective and X is compact, then f^{-1} is continuous on Y.*

Proof. Since f is bijective, $f^{-1} : Y \to X$ exists and also bijective. Let F be a closed set in X. Then $\left(f^{-1}\right)^{-1}(F) = f(F)$ and by Theorem 2.2, $f(F)$ is closed in Y. Thus, the inverse image of closed set is closed and hence f^{-1} is continuous. □

Remark 4.2. In Theorem 2.3, if X is not compact, then f^{-1} need not be continuous. For example, consider and identity function $I : (\mathbb{R}, d) \to (\mathbb{R}, \mathcal{U})$ from $\mathbb{R}$ with discrete metric to $\mathbb{R}$ with usual metric. Then I is continuous but I^{-1} is not.

4.3 Continuous Functions and Connected Sets

Theorem 4.11. *Let (X, d) and (Y, ρ) be metric spaces and let $f : X \to Y$ be a continuous functions. If C is connected subset of X, then $f(C)$ is connected subset of Y.*

Proof. Assume that $f(C)$ is disconnected. Then $f(C) = G \cup H$, where G and H are nonempty, disjoint open sets subsets of Y such that $f(C) \cap G$ and $f(C) \cap H$ are nonempty. Then

$$C \subseteq f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H).$$

Since f is continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are open in X. Moreover, $C \cap f^{-1}(G)$ and $C \cap f^{-1}(G)$ are nonempty and disjoint. It follows that C is disconnected, which is a contradiction. Hence $f(C)$ is connected. □

Theorem 4.12. *Let (X, d) be a connected metric space and $f : X \to \mathbb{R}$ be a continuous function. Then $f(X)$ is an interval.*

Proof. By theorem 3.1, $f(X)$ is connected subset of $\mathbb{R}$. Since "a subset of $\mathbb{R}$ is connected if and only if it is an interval" (Theorem 2.2). We have $f(X)$ is an interval. $\square$

Corollary 4.2. *[Intermediate Value Theorem] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $a, b, \in \mathbb{R}$ with $a < b$ and $f(a) \neq f(b)$. If α is a real numbers between $f(a)$ and $f(b)$, then there exists a real number c, $a \leq c \leq b$ such that $f(c) = \alpha$.*

Corollary 4.3. *A metric space (X, d) is disconnected if and only if there exists a continuous function $f : X \to \{0, 1\}$ form X onto the discrete two point space $\{0, 1\}$.*

Proof. Let X be disconnected. Then $X = A \cup B$, where A and B are nonempty, disjoint, open subsets of X. Define $f : X \to \{0, 1\}$ by

$$f(x) = \begin{cases} 0, & x \in A \\ 1, & x \in B. \end{cases}$$

Then, clearly f is continuous form X onto $\{0, 1\}$.

Conversely, assume that there exists a continuous function $f : X \to \{0, 1\}$ from X onto $\{0, 1\}$. Let X be connected. Then by Theorem 3.1, $f(X) = \{0, 1\}$ is connected, which is a contradiction. Hence X is disconnected. $\square$

4.4 Uniform Continuity

Before giving the definition of uniform continuity, we examine the following examples.

Consider a real-valued function $f : [-1, 1] \to \mathbb{R}$ defined as $f(x) = x^2$. Let x, x_0 be any points of $[-1, 1]$. Then

$$\begin{aligned} d\left(f(x), f(x_0)\right) &= |f(x) - f(x_0)| = \left|x^2 - x_0^2\right| \\ &= |x - x_0| \cdot |x + x_0| < \varepsilon \end{aligned}$$

whenever $|x - x_0| < \frac{1}{2}\varepsilon = \delta$, where δ is independent of the choice of x and x_0.

Thus for any $\varepsilon > 0$, there exists a $\delta = \frac{1}{2}\varepsilon$ such that for any $x, x_0 \in [-1, 1]$, we have

$$d\left(f(x), f(x_0)\right) < \varepsilon \quad \text{whenever} \quad d(x, x_0) < \delta.$$

Now, if we consider the same function $f(x) = x^2$ defined on $\mathbb{R}$, that is, $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Then for every real numbers x, x_0, we have

$$d\left(f(x), f(x_0)\right) = \left|x^2 - x_0^2\right| = |x - x_0| \cdot |x + x_0| < \varepsilon$$

whenever $|x - x_0| < \frac{\varepsilon}{|x+x_0|} = \delta$, where δ depends on ε and x_0.

In this way, we see that δ may depend not only on ε but also on x_0. Uniform continuity is essentially continuity plus the added condition that for each ε we can find a δ which works uniformly over the entire space, in the sense that it does not depends on x_0.

Definition 4.2. Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is said to be *uniformly continuous* if for each $\varepsilon > 0$, there exists a $\delta > 0$ (depends only on ε) such that for every $x_1, x_2 \in X$,

$$d(x_1, x_2) < \delta \quad \text{implies} \quad \rho\left(f(x_1), f(x_2)\right) < \varepsilon.$$

Remark 4.3. Every uniform continuous function is continuous but converse need not be true in general. For example, in the first example mentioned above, δ is independent of the choice of x and x_0, and therefore it is uniformity continuous. But in the later example, δ depends on ε and x_0, and hence it is only continuous but not uniformly continuous.

Theorem 4.13. *Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a continuous function. If X is compact then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$ and $x \in X$ be arbitrary. Consider the image $f(x)$ of x and the open sphere $S_\varepsilon(f(x))$. Since f is continuous, $f^{-1}\left(S_\varepsilon(f(x))\right)$ is an open set in X. Consider the family $\mathscr{F} = \left\{f^{-1}\left(S_\varepsilon(f(x))\right) : x \in X\right\}$ of these open sets in X. Then clearly $\mathscr{F}$ is an open cover of X. Since X is compact, it is sequentially compact and therefore, by Theorem .... there exists a Lebesgue number $\delta > 0$ for $\mathscr{F}$. Thus every open sphere of diameter less that δ will contain in at least one member of $\mathscr{F}$ and, consequently, we have

$$S_{\delta/2}(x) \subseteq f^{-1}\left(S_\varepsilon(f(x))\right) \quad \Rightarrow \quad f\left(S_{\delta/2}(x)\right) \subseteq S_\varepsilon(f(x)).$$

Hence for each $\varepsilon > 0$, there exists a $\tilde{\delta} > 0$ (independent of x) such that

$$d(x, y) < \frac{\delta}{2} = \tilde{\delta} \quad \Rightarrow \quad \rho\left(f(x), f(y)\right) < \varepsilon.$$

Hence f is uniformly continuous. □

Theorem 4.14. *Composition of two uniformly continuous functions is a uniformly continuous function.*

Theorem 4.15. *Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be an uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is also a Cauchy sequence in Y.*

Proof. Since f is uniformly continuous, for each $\varepsilon > 0$, there exists a $\delta > 0$ (only depends on ε) such that for all $x_1, x_2 \in X$,

$$d(x_1, x_2) < \delta \quad \text{implies} \quad \rho\left(f(x_1)), f(x_2)\right) < \varepsilon.$$

In particular, we have

$$d(x_n, x_m) < \delta \quad \text{implies} \quad \rho\left(f(x_n), f(x_m)\right) < \varepsilon \tag{$*$}$$

Since $\{x_n\}$ is a Cauchy sequence in X, for a given $\delta > 0$, there exists a positive integer N such that

$$d(x_n, x_m) < \delta \quad \text{for all } n, m \geq N \tag{$**$}$$

$(*)$ and $(**)$ imply that

$$\rho\left(f(x_n), f(x_m)\right) < \varepsilon \quad \text{for all } n, m \geq N.$$

Hence $\{f(x_n)\}$ is a cauchy sequence in Y. $\square$

Problem 4.3. *Give an example to show that the above theorem is not true if f is only continuous function.*

Problem 4.4. *Let (X, d) be a metric space and A be a subset of X. Prove that the function $f : X \to \mathbb{R}$ defined by*

$$f(x) = d(x, A) \quad \textit{for all } x \in X$$

is uniformly continuous.

Proof. By the triangular inequality

$$d(x, a) \leq d(x, y) + d(y, a) \quad \text{for all } a \in A, x \in X.$$

By taking infimum, we obtain

$$\inf_{a \in A} d(x, a) \leq d(x, y) + \inf_{a \in A} d(y, a).$$

Therefore

$$d(x, A) \leq d(x, y) + d(y, A)$$

and so

$$d(x, A) - d(y, A) \leq d(x, y) \quad \text{for all } x, y \in X.$$

By interchanging x and y, we obtain

$$d(y, A) - d(x, A) \leq d(y, x) = d(x, y).$$

Thus

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

Therefore, for a given $\varepsilon > 0$, choosing a δ such that $0 < \delta < \varepsilon$, we have

$$|f(x) - f(y)| = |d(x, A) - d(y, A)| \leq d(x, y) < \delta \leq \varepsilon,$$

that is,

$$|f(x) - f(y)| < \varepsilon \quad \text{whereever} \quad d(x, y) < \delta$$

Hence f is uniformly continuous on X. $\square$