King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Solution Math 202 Major Exam II
The First Semester of 2009-2010 (091)

Time Allowed: 90 Minutes

Name: ________________________ ID#: ________________

Section/Instructor: ________________ Serial #: ________________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		12
2		8
3		12
4		10
5		12
6		12
Total		66

Q:1 Consider the differential equation

$$y'' - 5y' + 6y = 3e^{4x}. \tag{1}$$

Let $y_1 = e^{2x}$ be a solution of the associated homogeneous equation of (1).

(a) (6 points) Use method of reduction of order to find a FIRST order SEPARABLE equation for the associated homogeneous equation of (1).

Sol: Let $y_2(x) = u(x)\, y_1(x)$ be the second solution of the associated homogeneous equation of (1).

Then $y_2' = u'y_1 + uy_1'$ and $y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$

Substituting $y_2,\ y_2'$, and y_2'' in the associated homogeneous equation of (1), we get,

$u''y_1 + 2u'y_1' + uy_1'' - 5u'y_1 - 5uy_1' + 6uy_1 = 0$

$\Rightarrow u''y_1 + 2u'y_1' - 5u'y_1 + u(y_1'' - 5y_1' + 6y_1) = 0$

$\Rightarrow u''e^{2x} + 2u'2e^{2x} - 5u'e^{2x} = 0$, since $y_1'' - 5y_1' + 6y_1 = 0$ and $y_1(x) = e^{2x}$.

$\Rightarrow (u'' + 4u' - 5u')\, e^{2x} = 0 \ \Rightarrow u'' - u' = 0$, since $e^{2x} \neq 0$

Let $u' = w$, then $u'' = w'$.

Substituting in $u'' - u' = 0$ we get the first order separable equation $w' - w = 0$.

(b) (3 points) Use the first order separable equation obtained in part (a) to find a second solution of the associated homogeneous equation of (1).

Sol: The first order separable equation $w' - w = 0$ can be written as $\dfrac{dw}{w} = dx$ whose solution is

$\ln|w| = x$ or $w(x) = e^x$.

Now $u'(x) = w(x) \ \Rightarrow u(x) = \int w(x)\, dx = \int e^x dx = e^x$

The second solution of the associated homogeneous equation is $y_2(x) = e^x e^{2x} = e^{3x}$

(c) (3 points) Find a particular solution of (1) and write its general solution.

Sol: Let $y_p = Ae^{4x}$ be the particular solution of (1).

Then substituting in (1), we get $16Ae^{4x} - 20Ae^{4x} + 6Ae^{4x} = 3e^{4x} \ \Rightarrow A = \dfrac{3}{2}$.

The general solution of (1) is $y = C_1e^{2x} + C_2e^{3x} + \dfrac{3}{2}e^{4x}$.

Q:2 (a) (3 points) The auxiliary equation of an 8^{th}-order linear homogeneous DE with real coefficients has the roots $m_1 = -3$ of multiplicity 1, $m_2 = 2$ of multiplicity 3, and $m_3 = 3+2i$ of multiplicity 2. Write the general solution of the DE.

Sol: The roots of the auxiliary equation are $m_1 = -3, m_2 = 2, 2, 2$ and $m_3 = 3+2i, 3+2i$.

Since the auxiliary equation has real coefficients, the other roots are $m = 3-2i, 3-2i$.

The solution of 8^{th} order DE is

$$y = C_1e^{-3x}+C_2e^{2x}+C_3xe^{2x}+C_4x^2e^{2x}+e^{3x}\left(C_5\cos(2x)+C_6\sin(2x)\right)+xe^{3x}\left(C_7\cos(2x)+C_8\sin(2x)\right)$$

(b) (5 points) Solve the following differential equation

$$y''' - y = 0.$$

Sol: The auxiliary equation of this equation is $m^3 - 1 = 0 \;\Rightarrow (m-1)\left(m^2+m+1\right) = 0$

The roots are $m = 1, \dfrac{1}{2} \pm \dfrac{\sqrt{3}i}{2}$

The solution is $y = C_1e^x + e^{-\frac{1}{2}x}\left(C_2\cos\left(\frac{\sqrt{3}}{2}x\right) + C_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

Q:3 (12 points) Use ANNIHILATOR approach to find the general solution of

$$y'' - 2y' + 5y = e^x\cos x + 2e^x\sin x.$$

Sol: Auxiliary equation of the associated homogeneous equation is

$m^2 - 2m + 5 = 0 \;\Rightarrow m = 1 \pm 2i.$

The complementary function is $y_c = e^x\left(C_1\cos(2x) + C_2\sin(2x)\right).$

Given equation can be written in operator for as $\left(D^2-2D+5\right)y = e^x\cos x + 2e^x\sin x$

Applying Annihilator $\left(D^2-2D+2\right)$ on both sides we get

$\left(D^2-2D+2\right)\left(D^2-2D+5\right)y = \left(D^2-2D+2\right)\left(e^x\cos x + 2e^x\sin x\right) = 0$

Auxiliary equation of this equation is $\left(m^2-2m+2\right)\left(m^2-2m+5\right) = 0$

$\Rightarrow m = 1 \pm i, 1 \pm 2i$

$\Rightarrow y = e^x\left(C_1\cos(2x) + C_2\sin(2x)\right) + e^x\left(C_3\cos(x) + C_4\sin(x)\right)$

Let $y_p = e^x (A\cos(x) + B\sin(x))$

then $y_p' = e^x (A\cos(x) + B\sin(x)) + e^x (-A\sin(x) + B\cos(x))$

and $y_p'' = e^x (A\cos(x) + B\sin(x)) + e^x (-A\sin(x) + B\cos(x))$
$\quad + e^x (-A\sin(x) + B\cos(x)) + e^x (-A\cos(x) - B\sin(x))$

$= 2e^x (-A\sin(x) + B\cos(x))$

Substituting in the given equation we get

$2e^x (-A\sin(x) + B\cos(x)) - 2e^x (A\cos(x) + B\sin(x))$

$-2e^x (-A\sin(x) + B\cos(x)) + 5e^x (A\cos(x) + B\sin(x)) = e^x \cos x + 2e^x \sin x$

$\Rightarrow 3e^x A\cos(x) + 3e^x C_4 \sin(x) = e^x \cos x + 2e^x \sin x \;\Rightarrow A = \frac{1}{3}$ and $B = \frac{2}{3}$.

The general solution is $y = y_c + y_p = e^x (C_1 \cos(2x) + C_2 \sin(2x)) + e^x \left(\frac{1}{3}\cos x + \frac{2}{3} e^x \sin x\right)$

Q:4 (10 points) Find the general solution of

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = x^{\frac{3}{2}},$$

given that $y_1 = x^{-\frac{1}{2}} \cos x$ is a solution of the corresponding homogeneous equation.

Sol: The given equation can be written as $y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = x^{-\frac{1}{2}}$

The second solution of the associated homogeneous equation is

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} = x^{-\frac{1}{2}} \cos x \int \frac{e^{-\int \frac{1}{x} dx}}{x^{-1}\cos^2 x} = x^{-\frac{1}{2}} \cos x \int \frac{e^{-\ln x}}{x^{-1}\cos^2 x} dx$$
$$= x^{-\frac{1}{2}} \cos x \int \frac{x^{-1}}{x^{-1}\cos^2 x} dx = x^{-\frac{1}{2}} \cos x \int \sec^2 x dx = x^{-\frac{1}{2}} \cos x \; \tan x = x^{-\frac{1}{2}} \sin x.$$

Now we find y_p using variation of parameters method

$$W = \begin{vmatrix} x^{-\frac{1}{2}} \cos x & x^{-\frac{1}{2}} \sin x \\ -\frac{1}{2} x^{-\frac{3}{2}} \cos x - x^{-\frac{1}{2}} \sin x & -\frac{1}{2} x^{-\frac{3}{2}} \sin x + x^{-\frac{1}{2}} \cos x \end{vmatrix} = x^{-1}$$

$$W_1 = \begin{vmatrix} 0 & x^{-\frac{1}{2}} \sin x \\ x^{-\frac{1}{2}} & -\frac{1}{2} x^{-\frac{3}{2}} \sin x + x^{-\frac{1}{2}} \cos x \end{vmatrix} = -x^{-1} \sin x$$

$$W_2 = \begin{vmatrix} x^{-\frac{1}{2}} \cos x & 0 \\ -\frac{1}{2} x^{-\frac{3}{2}} \cos x - x^{-\frac{1}{2}} \sin x & x^{-\frac{1}{2}} \end{vmatrix} = x^{-1} \cos x$$

$u_1(x) = \int \frac{W_1}{W} dx = \int -\sin x dx = \cos x$ and $u_2(x) = \int \frac{W_2}{W} dx = \int \cos dx = \sin x$

$y_p = u_1(x) y_1(x) + u_2(x) y_2(x) = x^{\frac{1}{2}} \cos^2 x + x^{\frac{1}{2}} \sin^2 x = x^{-\frac{1}{2}}$

The general solution is $y = C_1 x^{-\frac{1}{2}} \cos x + C_2 x^{-\frac{1}{2}} \sin x + x^{-\frac{1}{2}}$.

Q:5 (a) (8 points) Solve the following differential equation

$$2x^2 y'' + 7xy' + 3y = 2x^3.$$

Sol: To find y_c, let $y = x^m$, then $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$

Substituting in the associated homogeneous equation, we get the auxiliary equation

$$2m(m-1) + 7m + 3 = 0 \Rightarrow 2m^2 + 5m + 3 = 0 \Rightarrow (2m+3)(m+1) = 0 \Rightarrow m = -1, -\frac{3}{2}.$$

$y_c = C_1 x^{-1} + C_2 x^{-\frac{3}{2}}$. Let $y_1 = x^{-1}$ and $y_2 = x^{-\frac{3}{2}}$.

$$W = \begin{vmatrix} x^{-1} & x^{-\frac{3}{2}} \\ -x^{-2} & -\frac{3}{2}x^{-\frac{5}{2}} \end{vmatrix} = -\frac{1}{2}x^{-\frac{7}{2}},\ W_1 = \begin{vmatrix} 0 & x^{-\frac{3}{2}} \\ x & -\frac{3}{2}x^{-\frac{5}{2}} \end{vmatrix} = -x^{-\frac{1}{2}},\ W_2 = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & x \end{vmatrix} = 1$$

$u_1(x) = \int \frac{W_1}{W} dx = 2\int x^3 dx = \frac{1}{2}x^4$ and $u_2(x) = \int \frac{W_2}{W} dx = -2\int x^{\frac{7}{2}} dx = -\frac{4}{9}x^{\frac{9}{2}}$

$y_p = u_1(x) y_1(x) + u_2(x) y_2(x) = \frac{1}{2}x^3 - \frac{4}{9}x^3 = \frac{1}{18}x^3$

The general solution is $y = C_1 x^{-1} + C_2 x^{-\frac{3}{2}} + \frac{1}{18}x^3$.

(b) (4 points)Transform the equation in part (a) into an equation with constant coefficients.

Sol: Let $t = \ln x$, then using chain rule we can write

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{1}{x}$$

Substituting in the equation $2x^2y'' + 7xy' + 3y = 2x^3$, we get

$$2x^2\left(\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right) + 7x\frac{1}{x}\frac{dy}{dt} + 3y = 2e^{3t}$$

$$\Rightarrow 2\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 7\frac{dy}{dt} + 3y = 2e^{3t}$$

$$\Rightarrow 2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = 2e^{3t}$$

Q:6 (12 points) Use POWER SERIES method to solve the initial value problem

$$y'' - 2xy' + 8y = 0, \quad y(0) = 3, \quad y'(0) = 0.$$

Sol: Let $y = \sum_{n=0}^{\infty} c_n x^n$, then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2\sum_{n=1}^{\infty} n c_n x^n + 8\sum_{n=0}^{\infty} c_n x^n = 0$$

$$2c_2 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} - 2\sum_{n=1}^{\infty} n c_n x^n + 8c_0 + 8\sum_{n=1}^{\infty} c_n x^n = 0$$

$$2c_2 + 8c_0 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k - 2\sum_{k=1}^{\infty} k c_k x^k + 8\sum_{k=1}^{\infty} c_k x^k = 0$$

$$2c_2 + 8c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} x^k + (8-2k) c_k\right] x^k = 0$$

$$2c_2 + 8c_0 = 0 \;\Rightarrow c_2 = -4c_0 \text{ and } c_{k+2} = \frac{(2k-8)c_k}{(k+2)(k+1)}, \; k = 1, 2, 3, \ldots$$

$$k = 1, \; c_3 = \frac{-6}{3 \cdot 2} c_1 = -c_1$$

$$k = 2, \; c_4 = \frac{-4}{4 \cdot 3} c_2 = \frac{-1}{3}(-4c_0) = \frac{4c_0}{3}$$

$$k = 3, \; c_5 = \frac{-2}{5 \cdot 4} c_3 = \frac{1}{10} c_1$$

$k = 4, \; c_6 = \frac{0}{6 \cdot 5} c_4 = 0$. So $c_{2m} = 0$ for $m = 3, 4, \ldots$

$$y = c_0 + c_1 x + c_2 x^2 + + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 \ldots = c_0\left(1 - 4x^2 + \frac{4}{3}x^4\right) + c_1\left(x - x^3 + \frac{1}{10}x^5 + \ldots\right)$$

$$y' = c_0\left(-8x + \frac{16}{3}x^3\right) + c_1\left(1 - 3x^2 + \frac{1}{2}x^4 + \ldots\right)$$

$y(0) = 3 \;\Rightarrow c_0 = 3$ and $y'(0) = 0 \;\Rightarrow c_1 = 0$.

The solution is $y = 3\left(1 - 4x^2 + \frac{4}{3}x^4\right) = 3 - 12x^2 + 4x^4$.

Solution of Math 202 - 091 Exam 2 using Maple

```
> restart: with(DEtools):
```

Q:1

```
> Ode1:=diff(y(x),x$2)-5*diff(y(x),x$1)+6*y(x)=3*exp(4*x);
```

$$Ode1 := \left(\frac{d^2}{dx^2} y(x)\right) - 5\left(\frac{d}{dx} y(x)\right) + 6\, y(x) = 3\, e^{(4x)}$$

```
> dsolve(Ode1);
```

$$y(x) = e^{(2x)}\, \_C2 + e^{(3x)}\, \_C1 + \frac{3}{2} e^{(4x)}$$

Q:2(b)

```
> Ode2:=diff(y(x),x$3)-y(x)=0;
```

$$Ode2 := \left(\frac{d^3}{dx^3} y(x)\right) - y(x) = 0$$

```
> dsolve(Ode2);
```

$$y(x) = \_C1\, e^{x} + \_C2\, e^{\left(-\frac{x}{2}\right)} \sin\left(\frac{\sqrt{3}\, x}{2}\right) + \_C3\, e^{\left(-\frac{x}{2}\right)} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Q:3

```
> Ode3:=diff(y(x),x$2)-
2*diff(y(x),x$1)+5*y(x)=exp(x)*cos(x)+2*exp(x)*sin(x);
```

$$Ode3 := \left(\frac{d^2}{dx^2} y(x)\right) - 2\left(\frac{d}{dx} y(x)\right) + 5\, y(x) = e^{x} \cos(x) + 2\, e^{x} \sin(x)$$

```
> dsolve(Ode3);
```

$$y(x) = e^{x} \sin(2\, x)\, \_C2 + e^{x} \cos(2\, x)\, \_C1 + \frac{1}{3} e^{x} (\cos(x) + 2 \sin(x))$$

Q:4

```
> Ode4:=x^2*diff(y(x),x$2)+x*diff(y(x),x$1)+(x^2-
1/4)*y(x)=x^(3/2);
```

$$Ode4 := x^2 \left(\frac{d^2}{dx^2} y(x)\right) + x\left(\frac{d}{dx} y(x)\right) + \left(x^2 - \frac{1}{4}\right) y(x) = x^{(3/2)}$$

```
> dsolve(Ode4);
```

$$y(x) = \frac{\sin(x)\, \_C2}{\sqrt{x}} + \frac{\cos(x)\, \_C1}{\sqrt{x}} + \frac{1}{\sqrt{x}}$$

Q:5(a)

```
> Ode5:=2*x^2*diff(y(x),x$2)+7*x*diff(y(x),x$1)+3*y(x)=2*x^3;
```

$$Ode5 := 2x^2\left(\frac{d^2}{dx^2}y(x)\right)+7x\left(\frac{d}{dx}y(x)\right)+3y(x)=2x^3$$

```
> dsolve(Ode5);
```

$$y(x)=\frac{\_C2}{x^{(3/2)}}+\frac{\_C1}{x}+\frac{x^3}{18}$$

Q:6

```
> Ode6:=diff(y(x),x$2)-2*x*diff(y(x),x$1)+8*y(x)=0;
```

$$Ode6 := \left(\frac{d^2}{dx^2}y(x)\right)-2x\left(\frac{d}{dx}y(x)\right)+8y(x)=0$$

```
> dsolve({Ode6,y(0)=3,D(y)(0)=0});
```

$$y(x)=3-12x^2+4x^4$$