King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Solution Midterm Exam 1 for Math 201 (073)

Date: July 23, 2008 Instructor: Dr. Muhammad Yousuf

Q.1: Let parametric equations of a curve are: $x(t) = a(\cos t + t\sin t)$and $y(t) = a(\sin t - t\cos t)$.

(a) Find $\dfrac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t$

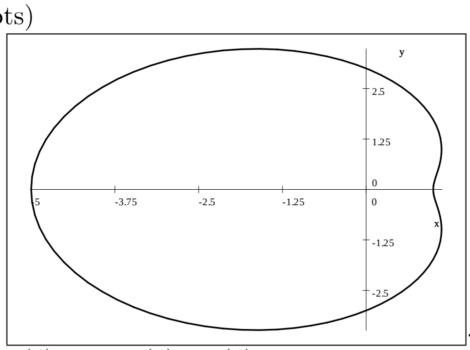
(b) Find $\dfrac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t$

(c) Find $\dfrac{dy}{dx} = \tan t$

(d) Find $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \dfrac{\sec^2 t}{at\cos t} = \dfrac{1}{at\cos^3 t}$

(e) Find length of the curve for $0 \le t \le 1$. $L = \int\limits_0^1 \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt = \int\limits_0^1 \sqrt{a^2t^2}\,dt = \dfrac{|a|}{2}.$

Q.2: Sketch graph of the polar curve $r = 3 - 2\cos(\theta)$. Show the points on the curve for $\theta = 0,\ \dfrac{\pi}{2},\ \pi,\ \dfrac{3\pi}{2}, 2\pi$.(4 pts)



. Also $x = r\cos(\theta) = 3\cos(\theta) - 2\cos^2(\theta)$, and $y = r\sin(\theta) = 3\sin(\theta) - 2\sin(\theta)\cos(\theta)$

(a) Find $\dfrac{dx}{d\theta} = -3\sin(\theta) + 4\cos(\theta)\sin(\theta)$

(b) Find $\dfrac{dy}{d\theta} = 3\cos(\theta) - 2\cos^2(\theta) + 2\sin^2(\theta)$.

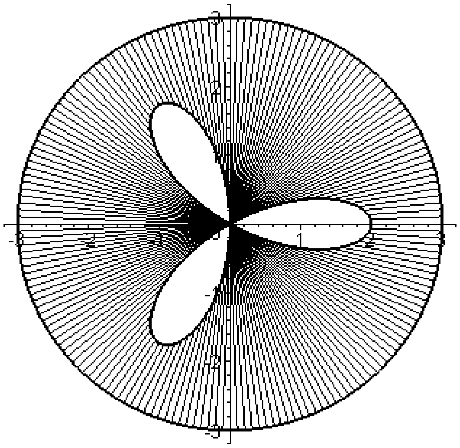
(c) Find $\dfrac{dy}{dx} = \dfrac{3\cos(\theta) - 2\cos(2\theta)}{-3\sin(\theta) + 2\sin(2\theta)}$

(d) Find equation of tangent line at $\theta = \dfrac{\pi}{4}$ (3 pts)

At $\theta = \dfrac{\pi}{4},\ x = 3\dfrac{\sqrt{2}}{2} - 1,\ y = 3\dfrac{\sqrt{2}}{2} - 1$, Slope $m = \dfrac{3\dfrac{\sqrt{2}}{2} - 0}{-3\dfrac{\sqrt{2}}{2} + 2}$

$$\left(y - \left(3\frac{\sqrt{2}}{2} - 1\right)\right) = \frac{3\dfrac{\sqrt{2}}{2}}{-3\dfrac{\sqrt{2}}{2} + 2}\left(x - \left(3\frac{\sqrt{2}}{2} - 1\right)\right)$$

Q.3: Find area of the region that lies inside $r = 3$ and outside $r = 2\cos(3\theta)$. (10 pts)



Area of three leaves is

$$A_1 = 6\,\frac{1}{2}\int_0^{\frac{\pi}{6}} (2\cos(3\theta))^2\,d\theta = 12\int_0^{\frac{\pi}{6}} \cos^2(3\theta)\,d\theta = \pi \text{ and area of circle is } A_2 = 9\pi.$$

So the required area is $A = A_2 - A_1 = 8\pi$.

Q.4: Let two forces $\tilde{\mathbf{F}}_1$ and $\tilde{\mathbf{F}}_2$ are acting at a point P and making angle 30^o and 135^o respectively with the horizontal line passing through the point P. If $\left\|\tilde{\mathbf{F}}_1\right\| = 10$ and $\left\|\tilde{\mathbf{F}}_2\right\| = 12$. Find the sum of the two forces. (10 pts)

$\mathbf{F}_1$=$\langle 10\cos 30^o, 10\sin 30^o\rangle = \langle 5\sqrt{3}, 5\rangle$ and $\mathbf{F}_2 = \langle 12\cos 135^o, 12\sin 135^o\rangle = \langle -6\sqrt{2}, 6\sqrt{2}\rangle$.

$\mathbf{F}_1$+F_2=$\langle 5\sqrt{3} - 6\sqrt{2}, 5 + 6\sqrt{2}\rangle$.

Q.5: Find equation of a sphere with end points of one of its diameter as $(2,1,4)$ and $(4,3,10)$. Also write its intersections with coordinate planes. (10 pts)

Center is $\left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right) = (3,2,7)$ and radius is $R = \sqrt{(3-2)^2 + (2-1)^2 + (7-4)^2} = \sqrt{11}$.

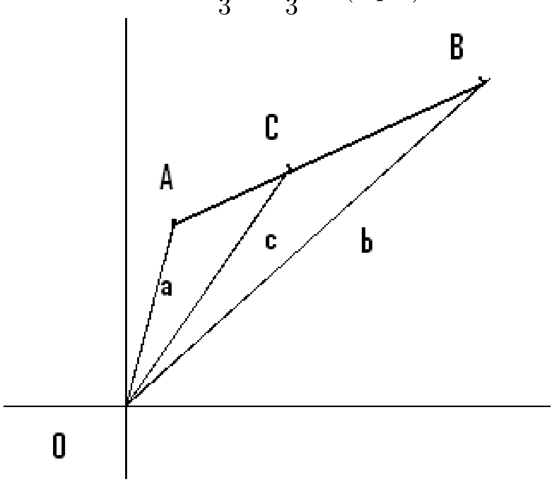
Equation of the sphere is $(x-3)^2 + (y-2)^2 + (z-7)^2 = 11$.

Intersection with $xy-plane$ $(z=0)$ is $(x-3)^2 + (y-2)^2 = 11 - 49 = -38$, so no intersection with $xy-plane$.

Intersection with $xz-plane$ $(y=0)$ is $(x-3)^2 + (z-7)^2 = 11 - 4 = 7$, a circle.

Intersection with $yz-plane$ $(x=0)$ is $(y-2)^2 + (z-7)^2 = 11 - 9 = 2$, a circle.

Q.6: Let C be a points on the line segment AB that is twice as far from B as it is from A. If $\tilde{\mathbf{a}}$ is the position vector of the points A, $\tilde{\mathbf{b}}$ is the position vector of the points B, and $\tilde{\mathbf{c}}$ is the position vector of the points C. Then show that $\tilde{\mathbf{c}} = \frac{2}{3}\tilde{\mathbf{a}} + \frac{1}{3}\tilde{\mathbf{b}}$. (9 pts)



$$\tilde{\mathbf{c}} = \tilde{\mathbf{a}} + AC = \tilde{\mathbf{a}} + \frac{1}{3}AB = \tilde{\mathbf{a}} + \frac{1}{3}\left(-\tilde{\mathbf{a}} + \tilde{\mathbf{b}}\right) = \frac{2}{3}\tilde{\mathbf{a}} + \frac{1}{3}\tilde{\mathbf{b}}.$$

Q.7: Show that the vector $\tilde{\mathbf{n}} = \langle a, b\rangle$ is perpendicular to the line $ax + by + c = 0$. (7 pts)

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the line $ax + by + c = 0$. Then the vector $PQ = \langle x_2 - x_1, y_2 - y_1\rangle$ is along the line and

$\tilde{\mathbf{n}} \cdot \mathbf{PQ} = a(x_2 - x_1) + b(y_2 - y_1) = ax_2 + by_2 - (ax_1 + by_1) = -c - (-c) = 0.$

So $\tilde{\mathbf{n}}$ is perpendicular to the line.

Q.8: Determine whether the four points $(1,1,1)$, $(2,4,6)$, $(3,-1,2)$, and $(6,2,8)$ lie in the same plane or not. (8 pts)

Let $A(1,1,1)$, $B(2,4,6)$, $C(3,-1,2)$, and $D(6,2,8)$. Consider the three vectors

$AB = \langle 1,3,5\rangle$, $AC = \langle 2,-2,1\rangle$, and $AD = \langle 5,1,7\rangle$. These three vectors are coplanar if all the four points lie in the same plane.

The three vectors are coplanar if $AB \cdot (AC \times AD) = 0$,

$AB \cdot (AC \times AD) = \begin{vmatrix} 1 & 3 & 5 \\ 2 & -2 & 1 \\ 5 & 1 & 7 \end{vmatrix} = 18 \neq 0$, so the points do not lie in the same plane.

Q.9: Find equation of a line containing the point $(2,3,-1)$ and is perpendicular to the plane $2x+3y-z=2$. Write your answer in symmetric and parametric form. (6 pts)

The direction vector of the line is $\mathbf{v} = \langle 2,3,-1\rangle$, and equation os the line is

$\dfrac{x-2}{2} = \dfrac{y-3}{3} = \dfrac{z+1}{-1} = t$

$x = 2t + 2,\ y = 3t + 3,\ z = -t - 1.$

Q.10: Find equation of a plane passing through the point $(1,2,3)$, and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$. (10 pts)

To find a point on the line of intersection, let $z = 0$, then $x = 1$ and $y = 1$, $(1,1,0)$

A vector parallel to the required plane is $\mathbf{v} = \langle 0,1,3\rangle$.

Also the vector $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{k} - 5\mathbf{j}$ is parallel to the required plane.

So normal to the require plane is $\mathbf{n} = \mathbf{v} \times (\mathbf{n_1} \times \mathbf{n_2}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 2 & -5 & -3 \end{vmatrix} = 12\mathbf{i} + 6\mathbf{j} - 2\mathbf{k} = 2(6\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

Equation of the required plane is $6(x-1) + 3(y-2) - 1(z-3) = 6x + 3y - z - 9 = 0$

OR $6(x-1) + 3(y-1) - 1(z-0) = 6x + 3y - z - 9 = 0$

Q.11: Find angle between the planes $2x - y + 3z = 1$ and $x + y - z = 2$. (5 pts)

$\mathbf{n}_1 = \langle 2,-1,3\rangle$ and $\mathbf{n_2} = \langle 1,1,-1\rangle$. If θ is the angle between two planes, then

$\cos(\theta) = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \dfrac{2-1-3}{\sqrt{14}\sqrt{3}} = \dfrac{-2}{\sqrt{42}}.$

Q.12: Find the distance of the points $(3,-2,7)$ from the plane $4x - 6y + z - 5 = 0$. (5 pts)

$d = \dfrac{|4(3) - 6(-2) + 1(7) - 5|}{\sqrt{16 + 36 + 1}} = \dfrac{26}{53}\sqrt{53}.$