Solution of Important Homework Problem

Q.34-(20.1): By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$,
therefore $(\cos\theta + i\sin\theta)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

Q.1-(20.2): $z = x + iy$
$|z - 8 + 4i| = |z - (8 - 4i)| = 9$ is a circle of radius 9 and center at $8 - 4i$.

Q.2-(20.2): $|z| = |x + iy| = \sqrt{(x^2 + y^2)}$ and $|z - i| = |x + iy - i| = \sqrt{(x^2 + y^2 - 2y + 1)}$
$x^2 + y^2 = x^2 + y^2 - 2y + 1$
$y = \frac{1}{2}$, which is a vertical line.

Q.16-(20.2): $1 < |z| \leq 4$ or $1 < x^2 + y^2 \leq 16$, is the annular region lying between two concentric circles of radius 1 and 4.

Q.37-(20.2): Find the limit of $\left\{\dfrac{1 + 3n^2 i}{2n^2 - n}\right\}$ as $n \to \infty$.

$$\liminf_{n\to\infty} \frac{1 + 3n^2 i}{2n^2 - n} = \liminf_{n\to\infty} \frac{1}{2n^2 - n} + i \liminf_{n\to\infty} \frac{3n^2}{2n^2 - n} = 0 + \tfrac{3}{2} i$$

Q.2-(21.1): $f(z) = z^2 - iz = x^2 + 2ixy - y^2 - ix + y = (x^2 - y^2 + y) + i(2xy - x)$
$u(x, y) = x^2 - y^2 + y$, $v(x, y) = 2xy - x$

$$\frac{\partial u(x,y)}{\partial x} = 2x,\ \frac{\partial u(x,y)}{\partial y} = -2y + 1,\ \frac{\partial v(x,y)}{\partial x} = 2y - 1,\ \frac{\partial v(x,y)}{\partial y} = 2x$$

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} = 2x, \text{ and } \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y} = 2y - 1$$

CR equations are satisfied every where, therefore f is differentiable every where.

Q.12-(21.1): $f(z) = \dfrac{z - i}{z + i} = \dfrac{x + iy - i}{x + iy + i} = \dfrac{x + i(y - 1)}{x + i(y + 1)} = \dfrac{x^2 + y^2 - 1 - 2xi}{x^2 + (y + 1)^2}$

$$u(x, y) = \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} \text{ ,and } v(x, y) = \frac{-2x}{x^2 + (y + 1)^2}$$

$$\frac{\partial u(x,y)}{\partial x} = 4x \frac{y + 1}{(x^2 + y^2 + 2y + 1)^2},\ \frac{\partial u(x,y)}{\partial y} = 2 \frac{y^2 + 2y - x^2 + 1}{(x^2 + y^2 + 2y + 1)^2},$$

$$\frac{\partial v(x,y)}{\partial x} = -2 \frac{y^2 + 2y - x^2 + 1}{(x^2 + y^2 + 2y + 1)^2},\ \frac{\partial v(x,y)}{\partial y} = 4x \frac{y + 1}{(x^2 + y^2 + 2y + 1)^2}$$

CR equations are satisfied and $u(x, y)$ and $v(x, y)$ are continuous every where except at

$x = 0, y = -1$ or $z = -i$. Thus $f(z)$ is differentiable at all points except at $z = -i$.

Q.3-(21.2): Determine radius of convergence of $\sum\limits_{n=0}^{\infty} \left(\dfrac{n^n}{(n + 1)^n}\right)(z - 1 + 3i)^n$

$$R = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+2)^{n+1}} (z-1+3i)^{n+1} \frac{(n+1)^n}{n^n} \frac{1}{(z-1+3i)^n} \right|$$
$$= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+2)^{n+1}} \frac{(n+1)^n}{n^n} (z-1+3i) \right|$$
$$= \lim_{n\to\infty} \left(\frac{n+1}{n+2} \right)^{n+1} \left(\frac{n+1}{n} \right)^n |(z-1+3i)|$$
$$= \lim_{n\to\infty} \left(\frac{n+1}{n+1+1} \right)^{n+1} \left(\frac{n+1}{n} \right)^n |(z-1+3i)|$$
$$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1+1}{n+1} \right)^{n+1}} \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n |(z-1+3i)|$$
$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n+1} \right)^{n+1}} \left(1+\frac{1}{n} \right)^n |(z-1+3i)|$$
$$= \frac{1}{e} e\, |(z-(1-3i))| = |(z-(1-3i))| < 1 \qquad \text{Using} \qquad \lim_{x\to\infty} \left(1+\frac{1}{x} \right)^x = e$$
$\Rightarrow$ Radius of convergence is 1 and center of convergence is $1-3i$.

Q.9-(21.2): Determine radius of convergence of $\sum\limits_{n=0}^{\infty} \left(\frac{e^{in}}{2n+1} \right) (z+4)^n$

$$R = \lim_{n\to\infty} \left| \left(\frac{e^{i(n+1)}}{2n+3} \right) (z+4)^{n+1} \frac{(2n+1)}{e^{in}} \frac{1}{(z+4)^n} \right|$$
$$= \lim_{n\to\infty} \left| \frac{(2n+1)}{(2n+3)} e^i (z+4) \right| = |z+4| < 1$$
$\Rightarrow$ Radius of convergence is 1 and center of convergence is $-4+0i$.

Q.2-(21.3): $\sin(1-4i) = \sin 1 \cosh 4 - i \cos 1 \sinh 4$

Q.4-(21.3): $\tan(0+3i) = i \tanh 3$

Q.6-(21.3): $\cot\left(1-i\frac{\pi}{4}\right) = \dfrac{\cos\left(1-i\frac{\pi}{4}\right)}{\sin\left(1-i\frac{\pi}{4}\right)}$

$$\cos\left(1-i\frac{\pi}{4}\right) = \cos 1 \cosh \tfrac{1}{4}\pi + i \sin 1 \sinh \tfrac{1}{4}\pi$$
$$\sin\left(1-i\frac{\pi}{4}\right) = \sin 1 \cosh \tfrac{1}{4}\pi - i \cos 1 \sinh \tfrac{1}{4}\pi$$
$$\cot\left(1-i\frac{\pi}{4}\right) = \frac{\cos\left(1-i\frac{\pi}{4}\right)}{\sin\left(1-i\frac{\pi}{4}\right)} = \frac{\cos 1 \cosh \frac{1}{4}\pi + i \sin 1 \sinh \frac{1}{4}\pi}{\sin 1 \cosh \frac{1}{4}\pi - i \cos 1 \sinh \frac{1}{4}\pi}$$
$$= \frac{\left(\cos 1 \cosh \frac{1}{4}\pi + i \sin 1 \sinh \frac{1}{4}\pi\right)\left(\sin 1 \cosh \frac{1}{4}\pi + i \cos 1 \sinh \frac{1}{4}\pi\right)}{\left(\sin 1 \cosh \frac{1}{4}\pi - i \cos 1 \sinh \frac{1}{4}\pi\right)\left(\sin 1 \cosh \frac{1}{4}\pi + i \cos 1 \sinh \frac{1}{4}\pi\right)}$$
$$= \frac{i \sinh \frac{1}{4}\pi \cosh \frac{1}{4}\pi + \sin 1 \cos 1}{\cosh^2 \frac{1}{4}\pi - \cos^2 1}$$

Q.8-(21.3): $\cos(2-i)-\sin(2-i)=\cos 2\cosh 1+i\sin 2\sinh 1-\sin 2\cosh 1+i\cos 2\sinh 1$

Q.11-(21.3): $z=x+iy,\ e^{z^2}=e^{x^2-y^2}e^{2(ixy)}=e^{x^2-y^2}\left(\cos(2xy)+i\sin(2xy)\right)$

$=e^{x^2-y^2}\cos(2xy)+ie^{x^2-y^2}\sin(2xy)$

$u(x,y)=e^{x^2-y^2}\cos(2xy),\ v(x,y)=e^{x^2-y^2}\sin(2xy)$

$$\frac{\partial u(x,y)}{\partial x}=2xe^{x^2-y^2}\cos 2xy-2e^{x^2-y^2}(\sin 2xy)\,y=2e^{x^2-y^2}[x\cos 2xy-y\sin 2xy]$$

$$\frac{\partial u(x,y)}{\partial y}=-2ye^{x^2-y^2}\cos 2xy-2e^{x^2-y^2}(\sin 2xy)\,x=-2e^{x^2-y^2}[y\cos 2xy+x\sin 2xy]$$

$$\frac{\partial v(x,y)}{\partial x}=2e^{x^2-y^2}(\sin 2xy)\,x+2ye^{x^2-y^2}\cos 2xy=2e^{x^2-y^2}[x\sin 2xy+y\cos 2xy]$$

$$\frac{\partial v(x,y)}{\partial y}=2xe^{x^2-y^2}\cos 2xy-2e^{x^2-y^2}(\sin 2xy)\,y=2e^{x^2-y^2}[x\cos 2xy-y\sin 2xy]$$

CR equations are satisfied.

Q.15-(21.3): $\sin^2(z)=\sin^2 x\cosh^2 y+2i\sin x\cosh y\cos x\sinh y-\cos^2 x\sinh^2 y$

$\cos^2(z)=\cos^2 x\cosh^2 y-2i\sin x\cosh y\cos x\sinh y-\sin^2 x\sinh^2 y$

$\sin^2(z)+\cos^2(z)=\sin^2 x\cosh^2 y-\cos^2 x\sinh^2 y+\cos^2 x\cosh^2 y-\sin^2 x\sinh^2 y=1$

Using $\sin^2(x)=1-\cos^2(x)$

Q.19-(21.3): $\sinh(z)=-i\sin(iz)$ and $\sin(w)=0$ iff $w=n\pi$, n any integer.

Since $i\neq 0$, therefore $\sinh(z)=0$ iff $iz=n\pi$ or $z=in\pi$, That is $y=n\pi$.

Q.23-(21.3): $e^z=-2\ \Rightarrow e^xe^{iy}=-2$ or $e^x\cos y+e^xi\sin y=-2$

Comparing real and imaginary parts $e^x\cos y=-2$ and $e^x\sin y=0$

$e^x\sin y=0\ \Rightarrow y=n\pi$, n any integer

If $y=2k\pi$, then $\cos(y)=1$ and we get $e^x=-2$, not possible

If $y=(2k+1)\pi$, then $\cos(y)=-1$, and we get $e^x=2\ \Rightarrow x=\ln(2)$.

Thus $x=\ln(2)$ and $y=(2k+1)\pi$.

Q.2-(21.5): $(1+i)^{2i}=e^{2i\log(1+i)}=e^{2i\left(\ln(2)+i\left(\frac{\pi}{4}+2n\pi\right)\right)}=e^{-\left(\frac{\pi}{2}+4n\pi\right)}[\cos(\ln 2)+i\sin(\ln 2)]$

Q.6-(21.5): $(1-i)^{\frac{1}{3}}=\sqrt{2}\left[e^{-i\left(\frac{\pi}{4}+2n\pi\right)}\right]^{\frac{1}{3}}=\sqrt[6]{2}e^{-i\left(\frac{\pi}{12}+\frac{2n\pi}{3}\right)}$, $n=0,1,2$.

Q.10-(21.5): $6^{(-2-3i)}=\frac{1}{36}e^{6n\pi}[\cos(3\ln 6)-i\sin(3\ln 6)]$

Q.4-(12.8): The boundary curve $\sum$ is the circle $x^2+y^2=6$ in the $xy-plane$ and is parametrized as

$x(t)=\sqrt{6}\cos(t),\ y(t)=\sqrt{6}\sin(t),\ z=0,\ 0\leq t\leq 2\pi$.

$F=0i+6\cos^2(t)\,j+6\sin^2(t)\,k$, and $R=\sqrt{6}\cos(t)\,i+\sqrt{6}\sin(t)\,j+0k$

and $dR=-\sqrt{6}\sin(t)\,i+\sqrt{6}\cos(t)\,j+0k$

Thus $\oint\limits_C F\cdot dR=\int\limits_0^{2\pi}6\sqrt{6}\cos^3(t)\,dt=0$.

Q.6-(12.8): The circulation of F is $\oint\limits_C F \cdot dR$. Let $\sum$ be the disk $0 \leq x^2 + y^2 \leq 1$ with the boundary C

given by $x(t) = \cos(t)$, $y(t) = \sin(t)$, $z = 0$, $0 \leq t \leq 2\pi$.
Then the normal to $\sum$ is the vector $N = k$. We have $\nabla \times F = -zaj + (2xy + 1)k$,
so $\nabla \times F \cdot N = (2xy + 1)$, and $d\sigma = dA$.

$$\oint\limits_C F \cdot dR = \iint\limits_\Sigma \nabla \times F \cdot N \, d\sigma = \iint\limits_D (2xy + 1) \, dA = \int\limits_0^{2\pi} \int\limits_0^1 (2\cos(t)\sin(t) + 1) \, r dr dt = 0 + \pi.$$

Q.14-(12.8): $F = [e^{xyz}(1 + xyz), x^2 z, x^2 y)]$
$\nabla \times F = [0, xye^{xyz}(1 + xyz) + xye^{xyz} - 2xy, 2xz - xze^{xyz}(1 + xyz) - xze^{xyz}] \neq 0$
Thus F is not conservative.

Q.20-(12.8): $F = [y - 4xz, x, 3z^2 - 2x^2]$, $(1, 1, 1)$ and $(3, 1, 4)$
$\nabla \times F = [0, 0, 0]$, thus F is conservative.
The potential function $\varphi(x, y, z)$ such that $\nabla\varphi = F$ is $\varphi(x, y, z) = xy - 2x^2 z + z^3$.
Thus $\oint\limits_C F \cdot dR = \varphi(3, 1, 4) - \varphi(1, 1, 1) = -5$.