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Math 202 - 092

Major Exam 2, Key Solution.

Question 1:

- $y'' - 2y' - 3y = 4e^x - 9 \Rightarrow (D^2 - 2D - 3)(y) = 4e^x - 9.$
- An annihilator for $4e^x$ is $(D-1) = L_1$
- An annihilator for -9 is $D = L_2$.
- An annihilator for -9 ... $L_2(4e^x) = 4e^x \neq 0$

$\rightarrow$ Since $L_1(-9) = 9 \neq 0$ and $L_2(4e^x) = 4e^x \neq 0$

an annihilator for $4e^x - 9$ is $L_1 L_2 = D(D-1)$.

- We obtain: $D(D-1)(D^2 - 2D - 3)(y) = 0$.
- The characteristic equation associated to this equation is $r(r-1)(r^2 - 2r - 3) = 0$.

So $r(r-1)(r^2 - 2r - 3) = 0 \Leftrightarrow r(r-1)(r+1)(r-3) = 0$.

- The roots are $r=0$, $r=1$, $r=-1$ and $r=3$ and all of them are simple root

$\rightarrow$ The general solution is $\boxed{y = \underbrace{c_1 + c_2 e^x}_{y_p} + \underbrace{c_3 e^{-x} + c_4 e^{3x}}_{y_c}}$

- As $y_p = c_1 + c_2 e^x$, Differentiate and substitute in the equation: $y_p' = c_2 e^x$, $y_p'' = c_2 e^x$.

Then $y_p'' - 2y_p' - 3y_p = 4e^x - 9$

$\Rightarrow c_2 e^x - 2c_2 e^x - 3c_1 - 3c_2 e^x = 4e^x - 9$. Then

$-3c_1 - 4c_2 e^x = 4e^x - 9$. So $\left\{\begin{array}{l} -3c_1 = -9 \\ -4c_2 = 4 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 3 \\ c_2 = -1 \end{array}$

The general solution is $\boxed{y = c_3 e^{-x} + c_4 e^{3x} + 3 - e^{-x}}$

Question 2:

- Step 1: Write the differential equation in the Standard form: $y'' + \frac{y'}{x} - \frac{4}{x^2} = \frac{1}{x^4}$.

- Step 2: $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = \frac{-4}{x}$.

- Step 3: By "Variation of Parameters method"
$y_p = u_1 y_1 + u_2 y_2$, where u_1 and u_2 are given by:

- $u_1' = \dfrac{-x^{-2}\cdot\left(\frac{1}{x^4}\right)}{-\frac{4}{x}} = \dfrac{-\frac{1}{x^6}}{-\frac{4}{x}} = \dfrac{x^{-5}}{4}$. Then $\boxed{u_1 = -\frac{x^{-4}}{16}}$.

- $u_2' = \dfrac{x^2 \cdot 1/x^4}{-\frac{4}{x}} = \dfrac{\frac{1}{x^2}}{-\frac{4}{x}} = -\dfrac{1}{4x}$. Then $\boxed{u_2 = -\frac{1}{4}\ln x}$

- So that $y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{16x^4}\cdot x^2 - \frac{1}{4}\ln x \cdot \frac{1}{x^2}$.

Thus $\boxed{y_p = -\frac{1}{16x^2} - \frac{\ln x}{4x^2}}$

Question 3: $\boxed{10/11}$

Let us use the method of reduction of order to find a Second Solution y_2 linearly independent to y_1.

- Recall the formula: $y_2 = y_1 \int \dfrac{e^{-\int P(x)dx}}{y_1^2} dx$.

- Step 1: Write the Diff. equation in the standard form:

$$y'' - \frac{2x}{1-x^2} y' + \frac{2}{1-x^2} y = 0.$$

- Step 2: $P(x) = -\dfrac{2x}{1-x^2} = \dfrac{(1-x^2)'}{1-x^2}$.

- Step 3: $\int P(x)dx = \int \dfrac{(1-x^2)'}{1-x^2} dx = \operatorname{Ln}(1-x^2)$.

- Then $e^{-\int P(x)dx} = e^{-\operatorname{Ln}(1-x^2)} = e^{\operatorname{Ln}(1-x^2)^{-1}} = (1-x^2)^{-1} = \dfrac{1}{1-x^2}$

- Step 4 $y_2 = x \int \dfrac{\frac{1}{1-x^2}}{x^2} dx = x \int \dfrac{1}{x^2(1-x^2)} dx$.

- Step 5 Find $\int \dfrac{dx}{x^2(1-x^2)}$ by Partial Fractions:

$$\frac{1}{x^2(1-x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} + \frac{D}{1+x}.$$

Then: $1 = Ax(1-x)(1+x) + B(1-x)(1+x) + Cx^2(1+x) + Dx^2(1-$

- For $x=0$, $\boxed{B=1}$
- For $x=1$, $2C = 1 \Longrightarrow \boxed{C = 1/2}$
- For $x=-1$, $2D = 1 \Longrightarrow \boxed{D = 1/2}$
- For $x=2$, $-6A + 3B + 12C - 4D = 1 \Longrightarrow \boxed{A=0}$

We obtain: $\dfrac{1}{x^2(1-x^2)} = \dfrac{1}{x(1-x)(1+x)} = \dfrac{1}{x^2} + \dfrac{1}{2(1-x)} + \dfrac{1}{2(1+x)}$

Now, $\int \frac{dx}{x^2(1-x^2)} = \int \left(\frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)} \right) dx$

$= -\frac{1}{x} - \frac{1}{2} \ln|1-x| + \frac{1}{2} \ln|1+x|$

$= -\frac{1}{x} + \frac{1}{2} \ln\left| \frac{1+x}{1-x} \right|.$

→ Finally: $y_2 = x \int \frac{dx}{x^2(1-x^2)} = x\left[-\frac{1}{x} + \frac{1}{2} \ln\left| \frac{1+x}{1-x} \right| \right]$

So $\boxed{y_2 = -1 + \frac{x}{2} \ln\left| \frac{1+x}{1-x} \right|}$

Other Method.

Set $y_2 = u y_1$. Then $y_2' = u' y_1 + u y_1'$

and $y_2'' = u'' y_1 + 2u'y_1' + u y_1''$.

→ Substitute y_2 in the differential equation, we obtain:

$(1-x^2)(u''y_1 + 2u'y_1' + uy_1'') - 2x(u'y_1 + uy_1') + 2uy_1 = 0$

So $(1-x^2)(u''y_1 + 2u'y_1') + (1-x^2)uy_1'' - 2xu'y_1 - 2xuy_1' + 2uy_1 = 0$

Then $(1-x^2)(u''y_1 + 2u'y_1') - 2xu'y_1 + (1-x^2)uy_1'' - 2xuy_1' + 2uy_1 = 0$

So $(1-x^2)(u''y_1 + 2u'y_1') - 2xu'y_1 + u\underbrace{\left[(1-x^2)y_1'' - 2xy_1' + 2y_1\right]}_{=0 \text{ because } y_1 \text{ is a Solution.}} = 0$

Then $(1-x^2)(u''y_1 + 2u'y_1') - 2xu'y_1 = 0.$

→ Now substitute $y_1 = x$, $y_1' = 1$, we obtain:

$(1-x^2)(xu'' + 2u') - 2x^2u' = 0$

$x(1-x^2)u'' + 2(1-x^2)u' - 2x^2u' = 0$

$x(1-x^2)u'' + 2u' - 2x^2u' - 2x^2u' = 0$

$x(1-x^2)U'' + 2(1-2x^2)U' = 0.$

So $\boxed{\dfrac{U''}{U'} = 2\,\dfrac{2x^2-1}{x(1-x^2)}} = 2\cdot\dfrac{2x^2-1}{x(1-x)(1+x)}$

which is a separable equation.

→ Integrate: $\ln|u'| = 2\displaystyle\int \frac{2x^2-1}{x(1-x)(1+x)}\,dx.$

→ Again, by Partial fractions:

$$\frac{2x^2-1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}.$$

So $2x^2-1 = A(1-x^2) + Bx(1+x) + Cx(1-x)$

$2x^2-1 = A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$

$2x^2-1 = A + (B+C)x + (-A+B-C)x^2.$

$$\left\{\begin{array}{l} 2 = -A+B-C \\ 0 = B+C \\ -1 = A \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} 2 = 1+2B \\ B = -C \\ A = -1 \end{array}\right. \Longrightarrow \left\{\begin{array}{l} B = 1/2 \\ C = -1/2 \\ A = -1. \end{array}\right.$$

Therefore: $\ln|u'| = 2\displaystyle\int \frac{2x^2-1}{x(1-x)(1+x)}\,dx = 2\int\left(\frac{-1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\right.$

- $\ln|u'| = 2\left[-\ln|x| - \frac{1}{2}\ln|1-x| - \frac{1}{2}\ln|1+x|\right]$
- $\ln|u'| = -2\ln|x| - \ln|1-x| - \ln|1+x|.$

$\ln|u'| = \ln\left|\dfrac{1}{x^2(1-x^2)}\right|$ and so $U' = \dfrac{1}{x^2(1-x^2)}$

Then $U' = \dfrac{1}{x^2(1-x)(1+x)} = \dfrac{1}{x^2} + \dfrac{1}{2(1-x)} + \dfrac{1}{2(1+x)}$

So $U = -\dfrac{1}{x} - \dfrac{1}{2}\ln|1-x| + \dfrac{1}{2}\ln|1+x| = -\dfrac{1}{x} + \dfrac{1}{2}\ln\left|\dfrac{1+x}{1-x}\right.$

Finally $\boxed{y_2 = Uy_1 = xU = -1 + \frac{x}{2}\ln\left|\frac{1+x}{1-x}\right|}$

Question 4:

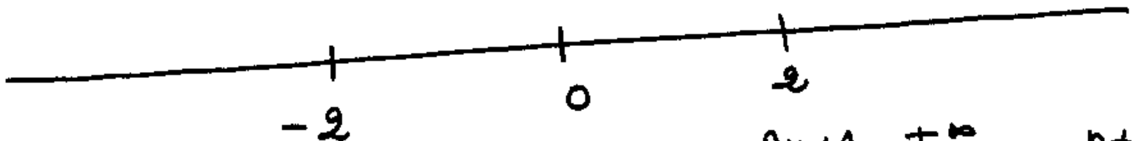
- Use the Ratio Test to find the radius and the interval of convergence for the power series $\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)}$.

Step 1: $a_n = \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)}$, $a_{n+1} = \frac{(-1)^{n+1} x^{2n+3}}{2^{2n+3}(2n+3)}$.

Step 2: $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{(-1)^{n+1}x^{2n+3}}{2^{2n+3}(2n+3)} \times \frac{2^{2n+1}(2n+1)}{(-1)^n x^{2n+1}}\right|$

$= \lim_{n\to+\infty} \left|\frac{x^2}{4}\,\frac{2n+1}{2n+3}\right| = \frac{x^2}{4} \underbrace{\lim_{n\to+\infty} \frac{2n+1}{2n+3}}_{=1} = \frac{x^2}{4}$.

Step 3 $\frac{x^2}{4} < 1 \Rightarrow x^2 < 4$
$\Rightarrow |x| < 2$. The radius of CV is $\boxed{R=2}$



Step 4: At $x_0 = -2$, $\sum_{n=0}^{+\infty} \frac{(-1)^n(-2)^{2n+1}}{2^{2n+1}(2n+1)} = \sum_{n=0} \frac{(-1)^{n+1}}{2n+1}$

which is <u>convergent by AST</u>.

Step 5: At $x_0 = 2$, $\sum_{n=0}^{+\infty} \frac{(-1)^n(+2)^{2n+1}}{2^{2n+1}(2n+1)} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}$

which is also convergent by AST.

Thus the interval of convergence is $\boxed{I = [-2, 2]}$

<u>Closed interval</u>

Question 5:

Code 01 $\longrightarrow$ d

Code 02 $\longrightarrow$ a

Code 03 $\longrightarrow$ b

Code 04 $\longrightarrow$ C

Question 6:

→ First Note that if r is a Complex root, then its conjugate $\bar{r}$ is also a root.

→ The roots of the differential equation would be $1, 1, 2, 1+i$ and $1-i$

Step 1: Find the characteristic (auxiliary) equation:

- $(r-1)(r-1)(r-2)(r-(1+i))(r-(1-i))=0$

So $(r-1)^2(r-2)(r-(1+i))(r-(1-i))=0$.

Then $(r^2-2r+1)(r-2)(r^2-2r+2)=0$

$(r^3-4r^2+5r-2)(r^2-2r+2)=0$

$r^5-2r^4+2r^3-4r^4+8r^3-8r^2+5r^3-10r^2+10r+$
$-2r^2+4r-4=0$.

Then $r^5-6r^4+15r^3-20r^2+14r-4=0$.

Thus the differential equation would be:

$$\boxed{y^{(5)}-6y^{(4)}+15y'''-20y''+14y'-4y=0}$$

Question 7:

The Corresponding homogeneous equation is:

$x^2y'' - 3xy' + 3y = 0$ which is a Cauchy-Euler equation:

Step 1: The auxiliary equation is: $m^2 - 4m + 3 = 0$

. obtained as follows: Set $y = x^m$. Then $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

. Substitute in the equation: $m(m-1)x^m - 3m\,x^m + 3x^m = 0$

So $[m(m-1) - 3m + 3]\,x^m = 0$. and so $m^2 - m - 3m + 3 = 0$

Then $m(m-1) - 3m + 3 = 0$ Hence $m^2 - 4m + 3 = 0$

$m^2 - m - 3m + 3 = 0$

$m(m-1) - 3(m-1) = 0$

. Step 2: The roots are $m = 1$, $m = 3$. $(m-1)(m-3) = 0$

. Step 3 The complementary solution is $y_c = c_1 x^1 + c_2 x^3$

Thus $\boxed{y_c = c_1 x + c_2 x^3}$ ie $\boxed{y_1 = x,\ y_2 = x^3}$

. Step 4 Use the Variation of parameters method to find y_p:

. $W(y_1, y_2) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3$.

. Equation in the Standard form: $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$

. $U_1' = \frac{-x^3 \cdot 2x^2e^x}{2x^3} = -x^2e^x$ and so $\boxed{U_1 = -x^2e^x + 2xe^x - 2e^x}$

. $U_2' = \frac{x \cdot 2x^2e^x}{2x^3} = e^x$ and so $\boxed{U_2 = e^x}$

. Step 5: $y_p = u_1y_1 + u_2y_2 = (-x^3 + 2x^2 - 2x)e^x + x^3e^x$

Thus $y_p = (2x^2 - 2x)e^x$.

Step 6 The general solution is $y = y_c + y_p$.

So $\boxed{y = c_1 x + c_2 x^3 + (2x^2 - 2x)e^x}$

- As $y_p = 2x^2 e^x - 2x e^x$
$= Ax^2 e^x + Bx e^x$

We obtain $A = 2$ and $B = -2$.

Thus $\boxed{2A + 3B = 4 - 6 = -2}$

Question 8:

- The characteristic (auxiliary) equation associated to the homogeneous equation is: $r^2+4r+4=0$.
- Then $(r+2)^2=0$ and so there ~~root~~ is only one double root $r=-2$.
- The general solution has the form:

$$y=y_c=(C_1+C_2x)e^{-2x}=C_1e^{-2x}+C_2xe^{-2x}.$$

- To solve the initial value problem, we have:

$$y(0)=1 \Longrightarrow \boxed{C_1=1}$$

- Now, $y'(x)=-2C_1e^{-2x}+C_2e^{-2x}-2C_2xe^{-2x}$

$$y'(0)=-1 \Longrightarrow -2C_1+C_2=-1.$$

Then $C_2=-1+2C_1=-1+2=1.$

Hence $\boxed{y=(1+x)e^{-2x}}$

Therefore $y(-\frac{1}{2})=(1-\frac{1}{2})e^{-2(-1/2)}$

$$\Longrightarrow y(-\tfrac{1}{2})=\tfrac{1}{2}e^1$$

$$\Longrightarrow \boxed{y(-\tfrac{1}{2})=\frac{e}{2}}$$

PART-II

Solutions

Question	Code 01	Code 02	Code 03	Code 04
4	a	d	c	d
5	d	a	b	c
6	a	d	b	c
7	b	c	d	e
8	d	c	b	a