KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematics and Statistics

MATH 202-(092)

Major Exam 1

Time: 120 Minutes

Name: Solution I.D. # Ser.#

Section: ___________

Show All Necessary Work

Calculators are not allowed in this exam

Question	Points
1	/6 3+3
2	/6
3	/13
4	/14
5	/14
6	/15 3+5+7
7	/10
8	/10 3+5+2
Total	/88

1. (a) Verify that $y(x) = \dfrac{-1}{x-c}$ is a one-parameter family of solutions of the D.E.

$$\frac{dy}{dx} - y^2 = 0$$

$$y = \frac{-1}{x-c} \Rightarrow y^2 = \left(\frac{-1}{x-c}\right)^2 = \frac{1}{(x-c)^2}$$

3

$$\frac{dy}{dx} = \frac{1}{(x-c)^2}$$

$$\therefore \frac{dy}{dx} - y^2 = \frac{1}{(x-c)^2} - \frac{1}{(x-c)^2} = 0$$

$\therefore$ $y(x) = \dfrac{-1}{x-c}$ is a one-parameter family of solutions.

(b) Find a singular solution for the above differential equation. Justify your answer.

Clearly $y=0$ is a solution of $\dfrac{dy}{dx} - y^2 = 0$.

3

This solution cannot be obtained from the solution (family) $y(x) = \dfrac{-1}{x-c}$, i.e. there is No Constant c that will give $0 = \dfrac{-1}{x-c}$. Hence $y=0$ is a Singular solution.

2. How many solutions are there to the initial value problem:

$$\frac{dy}{dx} + x^2y^2 = x, \qquad y(0) = 2. \qquad \text{Justify your answer.}$$

$$\frac{dy}{dx} = x - x^2y^2 = f(x,y)$$

6

$$\Rightarrow \frac{\partial f}{\partial y} = -2x^2y$$

Since $f(x,y)$ & $\dfrac{\partial f}{\partial y}$ are continuous everywhere, then the given IVP must have one and only one solution.

(13)

3. Solve the following differential equation by separating the variables:

$$\left(x^2+2x-xy-2y\right) e^{y-2x} dx + \left(y^3 - xy^2 + y - x\right) dy = 0$$

We observe that:

$$x^2+2x-xy-2y = x^2-xy+2x-2y = x(x-y)+2(x-y) = (x+2)(x-y)$$

Also,

$$y^3-xy^2+y-x = y^2(y-x)+(y-x) = (y^2+1)(y-x)$$

Thus, the above D.E. can be written as:

$$(x+2)(x-y)e^{y}e^{-2x}\,dx + (y^2+1)(y-x)\,dy = 0$$

$$(x+2)(x-y)e^{y}e^{-2x}\,dx = (y^2+1)(x-y)\,dy$$

$$(x+2)e^{-2x}\,dx = (y^2+1)e^{-y}\,dy \quad \text{(Separable)}$$

$$\Rightarrow \underbrace{\int (x+2)e^{-2x}\,dx}_{I} = \underbrace{\int (y^2+1)e^{-y}\,dy}_{J}$$

Integrating by parts:

$$I = -\tfrac{1}{2}(x+2)e^{-2x} + \tfrac{1}{2}\int e^{-2x}\,dx = -\tfrac{1}{2}(x+2)e^{-2x} - \tfrac{1}{4}e^{-2x} + C_1$$

$$= -\tfrac{1}{2}\left(x+\tfrac{5}{2}\right)e^{-2x} + C_1$$

$$J = \int (y^2+1)e^{-y}\,dy = -(y^2+1)e^{-y} + \int 2y e^{-y}\,dy$$

$$= -(y^2+1)e^{-y} - 2ye^{-y} + 2\int e^{-y}\,dy$$

$$= -(y^2+1)e^{-y} - 2ye^{-y} - 2e^{-y} + C_2$$

$$= -(y^2+2y+3)e^{-y} + C_2$$

∴ The solution of the given D.E. is:

$$\frac{1}{2}\left(x+\tfrac{5}{2}\right)e^{-2x} - (y^2+2y+3)e^{-y} = C$$

4. Solve the initial value problem:

$$(1-\ln x)\frac{dy}{dx}-\frac{y}{x}=1+\ln x\,, \qquad y(1)=2$$

14

$$\frac{dy}{dx}-\frac{1}{x(1-\ln x)}y=\frac{1+\ln x}{1-\ln x} \quad ----(*)$$

which is a linear D.E. in its standard form.

The integrating factor is:

$$\mu=e^{-\int\frac{dx}{x(1-\ln x)}}=e^{-\int\frac{\frac{1}{x}}{1-\ln x}dx}=e^{\ln|1-\ln x|}=1-\ln x\,,\ x\in(0,e)$$

Multiply both sides of $(*)$ by $1-\ln x$:

$$\frac{d}{dx}\left[y(1-\ln x)\right]=1+\ln x$$

$$y(1-\ln x)=\int(1+\ln x)dx$$

$$=x+x\ln x-x+c$$

$$=x\ln x+c$$

$$\Rightarrow\ y=\frac{x\ln x+c}{1-\ln x}$$

use the initial condition $y(1)=2$ to get $c=2$.

Hence, the solution is:

$$y=\frac{x\ln x+2}{1-\ln x}\,,\qquad x\in(0,e)$$

5. Use a suitable substitution to solve the following IVP:

14

$$(y-x)dy-\left(\frac{y^2}{x}+x\right)dx=0, \qquad y(1)=0$$

We can easily check that this is a homogeneous D.E.

Let $y=ux$. Then $dy=u\,dx+x\,du$.

Substitution gives:

$$(ux-x)(u\,dx+x\,du)-(u^2x+x)dx=0$$

$$(\cancel{u^2x}-ux-\cancel{u^2x}-x)dx+(ux^2-x^2)du=0$$

$$-x(u+1)dx+x^2(u-1)du=0$$

$$-\frac{dx}{x}+\frac{u-1}{u+1}du=0$$

$$\frac{u-1}{u+1}du=\frac{dx}{x}$$

$$\int\frac{dx}{x}=\int\frac{u-1}{u+1}du=\int\left[1-\frac{2}{u+1}\right]du$$

$$\ln|x|=u-2\ln|u+1|+\ln|c|$$

$$\ln|x|+2\ln|u+1|-\ln|c|=u$$

$$\ln|x|+2\ln\left|\frac{y}{x}+1\right|-\ln|c|=\frac{y}{x}$$

$$\ln|x|+\ln\left(\frac{x+y}{x}\right)^2-\ln|c|=\frac{y}{x}$$

$$\ln\left|\frac{(x+y)^2}{cx}\right|=\frac{y}{x}$$

$$\frac{(x+y)^2}{cx}=e^{\frac{y}{x}}$$

$y(1)=0 \Rightarrow c=1$

$\therefore$ The solution is: $\dfrac{(x+y)^2}{x}=e^{\frac{y}{x}}$

6. (a) Verify that the following differential equation is not exact:

$$(y+2x)dx + x(y+x+1)dy = 0$$

(3)

$M\,dx + N\,dy = 0 \Rightarrow M = y+2x, \quad N = x^2+xy+x$

$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 2x+y+1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ The D.E. is not exact.

(b) Find an integrating factor that will convert the above differential equation into an exact equation.

(5)

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{x^2+xy+x}\left[1-(2x+y+1)\right] = \frac{-2x-y}{x^2+xy+x} \quad !! \text{ function of } x \text{ and } y$$

$$\frac{1}{M}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{y+2x}\left[1-(2x+y+1)\right] = \frac{-2x-y}{2x+y} = -1 = g(y) \checkmark$$

$\therefore$ an integrating factor is $e^{-\int -1\,dy} = e^{\int dy} = e^{y}$

(c) Solve the above differential equation.

Multiply both sides of the given D.E. by e^y to get:

(7)

$$e^y(y+2x)dx + xe^y(y+x+1)dy = 0$$

$$\underbrace{(ye^y+2xe^y)}_{M}dx + \underbrace{(xye^y+x^2e^y+xe^y)}_{N}dy = 0 \quad [\text{Exact}]$$

$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = M \\ \frac{\partial f}{\partial y} = N \end{cases}$

$$f(x,y) = \int (ye^y+2xe^y)dx$$

$$= xye^y + x^2e^y + g(y)$$

$$\frac{\partial f}{\partial y} = xye^y + xe^y + x^2e^y + g'(y) = N = xye^y+x^2e^y+xe^y$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = K$$

$$\therefore f(x,y) = xye^y + x^2e^y + K$$

The solution is:

$$xye^y + x^2e^y = C$$

10

7. A metal bar initially at $60^0 F$ is placed in a freezer. The freezer is at the constant temperature $30^0 F$. Two minutes after, the temperature of the metal bar is $45^0 F$. What is the time needed for the temperature of the metal bar to reach $40^0 F$ in the freezer?

$T(t) :=$ temperature of the metal bar at time t

$T_m :=$ temperature of the freezer (constant).

By Newton's law of Cooling: $\frac{dT}{dt} = K(T - T_m)$

from which we have: $T(t) = T_m + Ce^{kt}$ --- (1)

Now, given: $\begin{cases} T(0) = 60 \\ T_m = 30 \\ T(2) = 45 \\ \text{need } t = ? \text{ if } T(t) = 40 \end{cases}$

From (1), we have

$T(t) = 30 + Ce^{kt}$ ---------- (2)

$T(0) = 60 \Rightarrow 30 + C = 60 \Rightarrow C = 30$

$\therefore$ (2) becomes:

$T(t) = 30 + 30e^{kt}$ -------- (3)

$T(2) = 45 \Rightarrow 30 + 30e^{2K} = 45$

$\Rightarrow e^{2K} = \frac{15}{30} = \frac{1}{2} \Rightarrow K = \frac{1}{2}\ln\frac{1}{2} = -\frac{\ln 2}{2}$

$\therefore$ (3) becomes:

$T(t) = 30 + 30e^{-\frac{\ln 2}{2}t}$

$T(t) = 40 \Rightarrow 30 + 30e^{-\frac{\ln 2}{2}t} = 40$

$30e^{-\frac{\ln 2}{2}t} = 10$

$\Rightarrow e^{-\frac{\ln 2}{2}t} = \frac{1}{3} \Rightarrow -\frac{\ln 2}{2}t = \ln\frac{1}{3} = -\ln 3$

$\Rightarrow t = 2\frac{\ln 3}{\ln 2}$ minutes

8. (a) Verify that $y_1 = 1+t$, $y_2 = 1+2t+t^2$ and $y_3 = 1-t^2$ are solutions of the differential equation $y''' = 0$ over $(-\infty,\infty)$.

(3)

$y_1 = 1+t \Rightarrow y_1' = 1 \Rightarrow y_1'' = y_1''' = 0 \Rightarrow y_1$ is a solution of $y''' = 0$

$y_2 = 1+2t+t^2 \Rightarrow y_2' = 2+2t \Rightarrow y_2'' = 2 \Rightarrow y_2''' = 0 \Rightarrow y_2$ is a solution of $y''' = 0$

$y_3 = 1-t^2 \Rightarrow y_3' = -2t \Rightarrow y_3'' = -2 \Rightarrow y_3''' = 0 \Rightarrow y_3$ " " "

Hence y_1, y_2 and y_3 are solutions of the D.E. $y''' = 0$.

(b) Determine whether y_1, y_2, y_3 form a fundamental set of solutions of the differential equation $y''' = 0$ over $(-\infty,\infty)$.

(5)

We use the Wronskian test:

$$W(y_1,y_2,y_3) = \begin{vmatrix} 1+t & 1+2t+t^2 & 1-t^2 \\ 1 & 2+2t & -2t \\ 0 & 2 & -2 \end{vmatrix} = (1+t)\begin{vmatrix} 2+2t & -2t \\ 2 & -2 \end{vmatrix} - \begin{vmatrix} 1+2t+t^2 & 1-t^2 \\ 2 & -2 \end{vmatrix}$$

$$= 0$$

Thus, y_1, y_2 and y_3 are not linearly independent and so they do not form a fundamental set of solutions.

(c) Can we form the general solution of the differential equation $y''' = 0$ from the solutions y_1, y_2 and y_3. Give reason.

(2)

No, we can not, because y_1, y_2 and y_3 do not form a fundamental set of solutions of the D.E. $y''' = 0$ as we have seen from part (b).