Math 202 Quiz # 6

Name: Solution **Section #** __________ **Sr. #**____

Find two power series solutions of the DE: $(x-1)y'' - x\,y' + y = 0$ about the ordinary point $x_0 = 0$.

Take $y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

substitute in the DE; $(x-1)y'' - xy' + y = 0$

$$(x-1)\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-1}}_{\text{put } k=n-1,\ n=k+1} - \underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} = 0$$

$$\sum_{k=1}^{\infty}(k+1)k\,c_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=1}^{\infty}(k+1)k\,c_{k+1}x^k - 2c_2 - \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} k c_k x^k + c_0 + \sum_{k=1}^{\infty} c_k x^k = 0$$

$$-2c_2 + c_0 + \sum_{k=1}^{\infty}\left[-(k+2)(k+1)c_{k+2} + (k+1)k\,c_{k+1} - (k-1)c_k\right]x^k = 0$$

$\Rightarrow -2c_2 + c_0 = 0$ & $-(k+2)(k+1)c_{k+2} + (k+1)k\,c_{k+1} - (k-1)c_k = 0$, $k \geq 1$

$\Rightarrow \boxed{c_2 = \frac{c_0}{2}}$ & $\boxed{c_{k+2} = \frac{k}{k+2}c_{k+1} - \frac{(k-1)}{(k+2)(k+1)}c_k}$, $k \geq 1$ Recurrence relation

$\underline{k}$ $\quad\underline{c_{k+2}}$

$k=1 \Rightarrow c_3 = \frac{1}{3}c_2 - 0c_1 = \frac{c_0}{6} = \frac{c_0}{3!}$ $\rightarrow$ c_1 arbitrary

$k=2 \Rightarrow c_4 = \frac{2}{4}c_3 - \frac{1}{4\cdot3}c_1 = \frac{1}{2}\cdot\frac{c_0}{6} - \frac{1}{12}\cdot\frac{c_0}{2} = \frac{c_0}{24} = \frac{c_0}{4!}$

$k=3 \Rightarrow c_5 = \ldots\ldots\ldots = \frac{c_0}{5!}$

$\vdots$

$\Longrightarrow c_n = \frac{c_0}{n!}$, $n \neq 1$ [c_1 arbitrary]

$\therefore$ The solution is $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$

$= c_0 + c_1 x + \frac{c_0}{2!}x^2 + \frac{c_0}{3!}x^3 + \frac{c_0}{4!}x^4 + \cdots$

$= c_1 x$ [illegible]