King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 202 Major Exam I
The Second Semester of 2008-2009 (082)

Time Allowed: 90 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

- All types of calculators are not allowed in this exam.
- Write clear steps.

KEY

Question #	Marks	Maximum Marks
1		6
2		6
3		6
4		7
5		6
6		7
7		6
Total		44

1. Consider the differential equation

$$\frac{dy}{dx} = y^2 \qquad \text{(A)}$$

(a) (2-points) Verify that $y = \dfrac{-1}{x+c}$ is a one-parameter family of solutions of equation (A).

$y = \dfrac{-1}{x+c}$

$\Rightarrow y' = \dfrac{1}{(x+c)^2} = y^2$ 2

(b) (2-points) Find the solution of equation (A) which satisfies the initial condition $y(0) = 2$.

$y(0) = 2 \Rightarrow 2 = -\dfrac{1}{c} \Rightarrow c = -\dfrac{1}{2}$ 1

$\Rightarrow y = -\dfrac{1}{x - 1/2}$ is solution of IVP 1

(c) (2-points) Find, if any, all singular solutions of equation (A) and determine the largest interval of existence of each singular solution obtained.

Singular solutions are given by solving 1

$y^2 = 0 \Rightarrow y = 0$

The largest interval of existence 1

is $(-\infty, \infty)$

2. (a) (3-points) Given that $x(t) = c_1 \cos wt + c_2 \sin wt$ is the general solution of the differential equation

$$\ddot{x} + w^2 x = 0 \quad \text{on} \quad (-\infty, \infty), \ w \neq 0.$$

Find all solutions which satisfy the boundary conditions $x(0) = 0$ and $\dot{x}\left(\frac{\pi}{2w}\right) = 0$.

$x(0) = 0 \Rightarrow c_1 = 0 \Rightarrow x(t) = c_2 \sin(wt)$ 1

$\dot{x}(t) = c_2 w \cos(wt)$

$\dot{x}\left(\frac{\pi}{2w}\right) = 0 \Rightarrow c_2 w \cos\frac{\pi}{2} = 0$ 1

$c_2 w \cos\frac{\pi}{2} = 0$ holds for all c_2

$\Rightarrow$

All solutions are $x(t) = c_2 \sin(wt)$ 1

(b) (3-points) Given $y_1 = e^x$ and $y_2 = e^x \tan x$ are two solutions of the differential equation

$$y'' - 2(1 + \tan x)y' + (1 + 2\tan x)y = 0. \qquad \text{(B)}$$

Determine whether or not the set $\{y_1, y_2\}$ forms a fundamental set of solutions of equation (B) on the interval $\left(0, \frac{\pi}{2}\right)$.

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^x \tan x \\ e^x & e^x(\tan x + \sec^2 x) \end{vmatrix}$$ 1

$$= e^{2x} \sec^2 x$$ 1

$$\neq 0 \quad \forall\, x \in \left(0, \frac{\pi}{2}\right)$$

$\Rightarrow$ solutions are independent and form a fundamental set of solutions 1

3. (a) (3-points) Find a suitable substitution that transforms the differential equation

$$xy\ dy + (4x^2 + y^2)\ dx = 0$$

to a separable equation. **Find the new equation, but do not find its solution.**

DE is homogeneous. 1

Let $y = ux$

$\Rightarrow dy = u\,dx + x\,du$

Substituting in DE

$\Rightarrow$

$ux\,du + (4 + 2u^2)\,dx = 0$

$\Rightarrow \dfrac{u}{4+2u^2}\,du = -\dfrac{dx}{x}$

which is separable

2

(b) (3-points) Find a suitable substitution that transforms the differential equation

$$(\cot x - y^2 \sin^3 x)\ dx + \frac{1}{y}\ dy = 0$$

to a linear equation. **Write the new equation in standard form, but do not find its solution.**

DE can be written as

$\dfrac{dy}{dx} + (\cot x)\,y = (\sin^3 x)\,y^3$ —— (*) 1

which is Bernoulli DE with $n = 3$

Let $u = y^{-2} \Rightarrow \dfrac{du}{dx} = -2y^{-3}\dfrac{dy}{dx}$

Substituting in (*)

$\Rightarrow -\dfrac{1}{2}\dfrac{du}{dx} + (\cot x)\,u = \sin^3 x$ 2

$\Rightarrow \dfrac{du}{dx} - 2(\cot x)\,u = -2\sin^3 x$

which is linear

4. (7-points) Solve the initial-value problem

$$(x \ln x)\frac{dy}{dx} + \cos^2 y = 1, \qquad y(e) = \frac{\pi}{4}.$$

$\Rightarrow$

$$x \ln x \frac{dy}{dx} = 1 - \cos^2 y$$

$$\Rightarrow \quad x \ln x \frac{dy}{dx} = \sin^2 y$$

$\Rightarrow$

$$\csc^2 y \, dy = \frac{1}{x \ln x} dx$$

3

$$\Rightarrow \quad -\cot y = \ln(\ln x) + C$$

2

$$y(e) = \frac{\pi}{4} \Rightarrow C = -1$$

$\Rightarrow$

$$\cot y + \ln(\ln x) = 1$$

is solution of IVP

2

5. (6-points) Find the general solution of the differential equation

$$y\,dx = (x + y\ln y)\,dy.$$

The equation is linear in x, $\frac{dx}{dy}$ because it can be written as

$$\frac{dx}{dy} - \frac{x}{y} = \ln y \quad \text{—— (A)}$$

2

Integrating factor is $u = e^{\int -\frac{1}{y}dy} = \frac{1}{y}$ 1

Multiplying (A) with $u = \frac{1}{y}$

$\Rightarrow$

$$\frac{d}{dy}\left(\frac{1}{y}x\right) = \frac{1}{y}\ln y$$

1

$$\Rightarrow \quad \frac{x}{y} = \int \frac{\ln y}{y}\,dy$$

$$= \frac{(\ln y)^2}{2} + C$$

2

$\Rightarrow$ General solution

$$x = \frac{y(\ln y)^2}{2} + Cy$$

6. (7-points) Solve the differential equation

$$(3x + 4y^2)dx + 4xy\, dy = 0,$$

by transforming it into an exact differential equation. [No other technique will be accepted].

$M(x,y) = 3x + 4y^2$ and $N(x,y) = 4xy$ (1)

$\Rightarrow M_y = 8y$ and $N_x = 4y$

- Finding integrating factor (2)

$\dfrac{M_y - N_x}{N} = \dfrac{1}{x}$, function of x alone

$\mu(x) = e^{\int \frac{dx}{x}} = x$

- Making DE exact (1)

$(3x^2 + 4xy^2)\,dx + 4x^2y\,dy = 0$

Solving $\dfrac{\partial f}{\partial x} = 3x^2 + 4xy^2$ (1)

$\Rightarrow f(x,y) = x^3 + 2x^2y^2 + g(y)$ —(*)

$\dfrac{\partial f}{\partial y} = 4x^2y + g'(y) = \mu(x)\,N(x,y) = 4x^2y$ (1)

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$

$\Rightarrow f(x,y) = x^3 + 2x^2y^2 + C_1$ (1)

$\Rightarrow x^3 + 2x^2y^2 = C$ is general solution.

7. (6-points) A glass of water initially at 50°F is placed in a freezer. The freezer is kept at the constant temperature of 30°F. After one hour the temperature of the water in glass is 40°F. Find the exact time needed for the temperature of the water to reach 32°F after it is placed in the freezer.

To solve IVP

$$\frac{dT}{dt} = k(T-30), \quad T(0) = 50$$ (1)

$$\Rightarrow \quad T(t) = 30 + Ce^{kt}$$ (1)

Using $T(0) = 50 \Rightarrow C = 20$

$$\Rightarrow \quad T(t) = 30 + 20e^{kt}$$ (1)

Finding k, using $T(1) = 40$

$$\Rightarrow \quad 40 = 30 + 20e^{k}$$

$$\Rightarrow \quad k = -\ln 2$$

$$\cancel{\Rightarrow T(t) = 30 + 20e^{-\ln 2}}$$

$$\Rightarrow \quad T(t) = 30 + 20e^{-t\ln 2} = 30 + 20(2^{-t})$$ (1)

Finding 't' such that $T(t) = 32$

$$\Rightarrow \quad 32 = 30 + 20(2^{-t})$$

$$\Rightarrow \quad 2^{-t} = \frac{1}{10}$$

$$\Rightarrow \quad t = \frac{\ln 10}{\ln 2}$$ (2)