King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 260
Exam II, Semester I, 2008-2009

Duration: 90 minutes

Name: Solution
ID: Ser.#
Section:

Q#	Marks	Maximum Marks
1		10
2		6
3		8
4		6
5		7
6		6
7		7
Total		50

1. Write clearly.
2. Show all your steps.
3. Calculators and mobile phones are NOT allowed in this exam.

1. (a) **Define** the Dimension of a vector space. Give an **example** of this definition.

The dimension of a vector space V is the number of vectors in a basis of V. For example, the vector space $\mathbb{R}^2$ has dimension 2 because $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$, where $e_1 = (1,0)$, $e_2 = (0,1)$.

(b) Let V be the vector space of all $n \times n$ matrices. A matrix A in V is called idempotent if $A^2 = A$. Prove or disprove that the set of all idempotent matrices form a subspace of V.

Let W be the subset of V consisting of all idempotent matrices, i.e. $W = \{A \in V : A^2 = A\}$.

Let $A, B \in W$. So, $A^2 = A$, $B^2 = B$.

Then
$$(A+B)^2 = A^2 + AB + BA + B^2$$
$$= A + AB + BA + B \neq A + B$$

$\Rightarrow$ $(A+B)^2 \neq A+B$

$\Rightarrow$ $A+B \notin W$

$\Rightarrow$ W is not a subspace of V

$\Rightarrow$ The statement is false.

(c) Let $C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} -g & -h & -i \\ 6d & 6e & 6f \\ 2a & 2b & 2c \end{bmatrix}$.

If $det(C) = \frac{-3}{4}$, find $det(B)$.

$$det(B) = \begin{vmatrix} -g & -h & -i \\ 6d & 6e & 6f \\ 2a & 2b & 2c \end{vmatrix} = 2\times 6 \begin{vmatrix} -g & -h & -i \\ d & e & f \\ a & b & c \end{vmatrix}$$

$$= 2\times 6\times(-1) \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = 2\times 6\times(-1)(-1) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= 12\, det(C) = 12 \times \left(\frac{-3}{4}\right) = -9$$

$\therefore \boxed{det(B) = -9}$

(d) Let $A = \begin{bmatrix} -3 & x & 0 \\ 5 & 7 & -9 \\ 0 & y & 11 \end{bmatrix}$.

If $A = A^T$, find the values of x and y.

$$A = A^T \Rightarrow \begin{bmatrix} -3 & x & 0 \\ 5 & 7 & -9 \\ 0 & y & 11 \end{bmatrix} = \begin{bmatrix} -3 & 5 & 0 \\ x & 7 & y \\ 0 & -9 & 11 \end{bmatrix}$$

$$\Rightarrow \boxed{x = 5}, \quad \boxed{y = -9}$$

2. Use Cramer's rule to find the value of x_2 for the system:

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 4 \\ x_1 + x_2 - 5x_3 &= -3 \\ -2x_1 - x_2 + 4x_3 &= 0 \end{aligned}$$

$$\Delta = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 1 & -5 \\ -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 9 \\ 1 & 1 & -5 \\ -2 & -1 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -5 \\ -1 & 4 \end{vmatrix} + 9 \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = -1 + 9 = 8$$

$$\Delta x_2 = \begin{vmatrix} 3 & 4 & -1 \\ 1 & -3 & -5 \\ -2 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ -3 & -5 \end{vmatrix} + 4 \begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix}$$

$$= -2(-23) - 52 = -6$$

$$\therefore \quad x_2 = \frac{\Delta x_2}{\Delta} = \frac{-6}{8} = \frac{-3}{4}$$

3. Let $S = \{v_1, v_2, v_3\}$, where $v_1 = (1,-1,2)$, $v_2 = (3,0,1)$, $v_3 = (1, -2, 2)$ and let $w = (2,-7,9)$.

   (a) Write, if possible, the vector w as a linear combination of the vectors v_1, v_2 and v_3.

$$W = C_1 V_1 + C_2 V_2 + C_3 V_3$$

$$(2,-7,9) = C_1(1,-1,2) + C_2(3,0,1) + C_3(1,-2,2)$$

$$\Rightarrow \left\{ \begin{array}{rl} C_1 + 3C_2 + C_3 &= 2 \\ -C_1 \quad\quad - 2C_3 &= -7 \\ 2C_1 + C_2 + 2C_3 &= 9 \end{array} \right\} \text{ solving this system:}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ -1 & 0 & -2 & -7 \\ 2 & 1 & 2 & 9 \end{array}\right] \xrightarrow[-2R_1+R_3]{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 3 & -1 & -5 \\ 0 & -5 & 0 & 5 \end{array}\right] \xrightarrow{-\frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 3 & -1 & -5 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & -1 & -5 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -2 \end{array}\right] \xrightarrow[-3R_2+R_1]{-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{l} C_3 = 2 \\ C_2 = -1 \\ C_1 = 3 \end{array}$$

$$\therefore \; W = 3V_1 - V_2 + 2V_3$$

   (b) Is $w \in Span(S)$? why ?

Yes, this is from (a), since W is a linear combination of V_1, V_2, V_3.

(c) Is S linearly dependent? verify ?

Calculate the determinant $\begin{vmatrix} v_1 & v_2 & v_3 \end{vmatrix}$:

$$\begin{vmatrix} 1 & 3 & 1 \\ -1 & 0 & -2 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 3 & -1 \\ -1 & 0 & -2 \\ 2 & 1 & 2 \end{vmatrix} = -3\begin{vmatrix} -1 & -2 \\ 2 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= -3(2) + 1$$

$$= -5$$

$$\neq 0$$

Hence v_1, v_2, v_3 are linearly independent.

i.e. S is linearly independent.

Another way: in part (a), if you solve the system using Gauss reduction you can notice that the coefficient matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & -2 \\ 2 & 1 & 2 \end{bmatrix} \sim I$. So A is invertible $\Rightarrow \det A \neq 0$
$\Rightarrow S$ is L. indep.

(d) Write the vector w as a linear combination of the standard basis.

$$w = (2, -7, 9) = 2(1,0,0) - 7(0,1,0) + 9(0,0,1)$$

$$\therefore w = 2e_1 - 7e_2 + 9e_3$$

4. Show that the matrix $A = \begin{bmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible; then find A^{-1}.

$$|A| = \cos\theta \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \cos^2\theta + \sin^2\theta = 1$$

$|A| \neq 0 \Rightarrow A$ is invertible.

<u>To find A^{-1}:</u>

$$A_{11} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta \qquad A_{21} = -\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -\sin\theta \qquad A_{31} = \begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} = 0$$

$$A_{12} = -\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = \sin\theta \qquad A_{22} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta \qquad A_{32} = -\begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & 0 \end{vmatrix} = 0$$

$$A_{13} = \begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} = 0 \qquad A_{23} = -\begin{vmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{vmatrix} = 0 \qquad A_{33} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

$$[A_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{adj } A = [A_{ij}]^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{\text{adj } A}{1} = \text{adj } A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Find a basis and the dimension of the solution space of the system:

$$\begin{aligned} 2x_1 + 7x_2 + 17x_3 + 19x_4 &= 0 \\ 2x_1 + 5x_2 + 11x_3 + 12x_4 &= 0 \\ x_1 + 5x_2 + 13x_3 + 14x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 2 & 7 & 17 & 19 & 0 \\ 2 & 5 & 11 & 12 & 0 \\ 1 & 5 & 13 & 14 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 5 & 13 & 14 & 0 \\ 2 & 5 & 11 & 12 & 0 \\ 2 & 7 & 17 & 19 & 0 \end{array}\right]$$

$$\xrightarrow[-2R_1 + R_3]{-2R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 5 & 13 & 14 & 0 \\ 0 & -5 & -15 & -16 & 0 \\ 0 & -3 & -9 & -9 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3} R_3} \left[\begin{array}{cccc|c} 1 & 5 & 13 & 14 & 0 \\ 0 & -5 & -15 & -16 & 0 \\ 0 & 1 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[-5R_3 + R_1]{5R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$\Rightarrow \quad x_4 = 0 \,,\quad x_3 = t \,,\quad x_2 = -3t \,,\quad x_1 = 2t$

The solution of the system is $X = \begin{bmatrix} 2t \\ -3t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

Hence $\{(2, -3, 1, 0)\}$ is a basis for the solution space,

and the dimension is 1.

6. Consider the following DE: $2y'' + 8y' + 26y = 0$.

(a) Write the characteristic equation of this DE.

$$2\lambda^2 + 8\lambda + 26 = 0$$

(b) Find the general solution of this DE.

$$2\lambda^2 + 8\lambda + 26 = 0$$

$$\Rightarrow \quad \lambda^2 + 4\lambda + 13 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2}$$

$$\lambda = -2 \pm 3i$$

The general solution is

$$y = e^{-2x}\left(C_1 \cos 3x + C_2 \sin 3x\right)$$

7. Solve the initial value problem:

$$y'' - 4y' - 5y = 0 \qquad y(1) = 0,\ y'(1) = 2.$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

$$\Rightarrow \lambda_1 = 5, \quad \lambda_2 = -1$$

The general solution is

$$y = c_1 e^{5x} + c_2 e^{-x}$$

$$y' = 5c_1 e^{5x} - c_2 e^{-x}$$

$$y(1) = 0 \Rightarrow c_1 e^{5} + c_2 e^{-1} = 0$$

$$y'(1) = 2 \Rightarrow 5c_1 e^{5} - c_2 e^{-1} = 2$$

solving

$$6c_1 e^{5} = 2 \Rightarrow c_1 = \tfrac{1}{3} e^{-5}$$

$$\therefore \tfrac{1}{3} e^{-5} e^{5} + c_2 e^{-1} = 0 \Rightarrow c_2 = -\tfrac{1}{3} e$$

$\therefore$ the solution of the given IVP is

$$y = \tfrac{1}{3} e^{-5} e^{5x} - \tfrac{1}{3} e e^{-x}$$

i.e. $$y = \tfrac{1}{3} e^{5x-5} - \tfrac{1}{3} e^{1-x}$$