King Fahd Univesity of Petroleum & Minerals
Department of Mathematical Sciences

MATH 101-051 (Midterm Exam)
November 27, 2005

Name: Solution ID:__________ Sec. #:____ Serial #______

**Time:100 minutes** **Max Points: 100**

**Show all necessary work**

**No calculator is allowed in the exam.**

**Points: ________/100**

**Part (I)**
**(Each question carries 5 pts)**

**Q.1. Evaluate** $\lim\limits_{t\to 2^-}\dfrac{|2t-4|}{2-t}$

$$= \lim_{t\to 2^-} \frac{-2t+4}{2-t}$$

$$= \lim_{t\to 2^-} \frac{2\cancel{(2-t)}}{\cancel{2-t}}$$

$$= 2$$

**Q.2. Evaluate** $\lim\limits_{y\to -\infty}\left(1+20y^3-12y^6+5000y^5\right)$.

$$= \lim_{y\to -\infty} (-12y^6) = -\infty$$

**Q.3. Evaluate** $\lim\limits_{w\to 0}\dfrac{1-\cos w}{\sin w}$

$$= \lim_{w\to 0} \frac{1-\cos w}{\sin w} \cdot \frac{1+\cos w}{1+\cos w}$$

$$= \lim_{w\to 0} \frac{1-\cos^2 w}{\sin w(1+\cos w)}$$

$$= \lim_{w\to 0} \frac{\sin^2 w}{\sin w\,(1+\cos w)}$$

$$= \lim_{w\to 0} \frac{\sin w}{1+\cos w} = \frac{0}{1+1} = 0$$

**Q.4.** Only **using the idea of derivative**, find

$$\lim_{t\to 0}\frac{\cos(y+t)-\cos y}{t}$$

$$= \frac{d}{dx}[\cos y] = -\sin y$$

**Part (II)**
**(Each question carries 7 pts)**

**Q.5**. Find a value of $\delta > 0$ which satisfies **the** $(\varepsilon - \delta)$ **definition of limit** when

$$\lim_{u \to -1/2} \frac{4u^2 - 1}{2u + 1} = -2; \ \varepsilon = 0.5 .$$

For the definition, we have

$$0 < |u + \tfrac{1}{2}| < \delta \Rightarrow \left| \frac{4u^2 - 1}{2u+1} + 2 \right| < 0.5$$

Now,

$$\left| \frac{4u^2-1}{2u+1} + 2 \right| = \left| \frac{\cancel{(2u+1)}(2u-1)}{\cancel{2u+1}} + 2 \right|$$

$$= |2u - 1 + 2| = |2u+1|$$
$$= 2|u + \tfrac{1}{2}|$$

So,

$$2|u + \tfrac{1}{2}| < 0.5$$
$$\Rightarrow |u + \tfrac{1}{2}| < 0.25$$
$$\therefore \delta = 0.25$$

**Q.6**. Find all **Removable Discontinuity(ies)** for the function $f(x) = \dfrac{x-2}{(x+1)(x^2 - 3x + 2)}$.

$$f(x) = \frac{x-2}{(x+1)(x-1)(x-2)}$$

f is not defined at $x = 1, -1, 2$.

$\boxed{x = 1}$:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x-2}{(x+1)(x-1)(x-2)} \quad \text{DNE}$$

$\boxed{x = -1}$

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x-2}{(x+1)(x-1)(x-2)} \quad \text{DNE}$$

$\boxed{x = 2}$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{\cancel{x-2}}{(x+1)(x-1)\cancel{(x-2)}}$$
$$= \lim_{x \to 2} \frac{1}{(x+1)(x-1)} = \frac{1}{3}$$

$\therefore$ f has a removable discontinuity at $x = 2$.

**Q.7.** Using the **definition of derivative**, find $g'(1)$ when $g(v) = \frac{1}{v^2}$.

$$g'(1) = \lim_{h\to 0} \frac{g(1+h) - g(1)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - 1}{h}$$

$$= \lim_{h\to 0} \frac{1-(1+h)^2}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{1-(1+2h+h^2)}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{-2h-h^2}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-2-h)}{\cancel{h}(1+h)^2}$$

$$= \lim_{h\to 0} \frac{-2-h}{(1+h)^2}$$

$$= -2$$

**Q.8.** Find the **equation of tangent line** to the graph of $f(x) = \frac{x}{1+x}$ at $x = 1$.

$$f'(x) = \frac{(1+x)-x}{(1+x)^2} = \frac{1}{(1+x)^2}$$

The slope of the tangent at $x=1$ is $f'(1) = \frac{1}{4}$

$x=1 \Rightarrow y = \frac{1}{2}$ $\therefore$ the tangent passes through $(1, \frac{1}{2})$

The equation of the tangent is

$$y - \frac{1}{2} = \frac{1}{4}(x-1)$$

i.e. $4y - x = 1$

**Q.9. Find** $\frac{d}{dx}\sqrt{\cos(\pi x^2 - 5)}$.

$$= \frac{1}{2\sqrt{\cos(\pi x^2-5)}} \cdot \left(-\sin(\pi x^2-5)\right) \cdot 2\pi x$$

$$= \frac{-\pi x \sin(\pi x^2-5)}{\sqrt{\cos(\pi x^2-5)}}$$

**Q.10.** A **particle** is moving along a straight line such that its **position** is given by $s(t)=5t^2-t+5$.

**a)** Find the **average velocity** of the particle in the time interval [1,3].

$$V_{ave} = \frac{S(3)-S(1)}{3-1}$$

$$= \frac{5(3)^2-3+5-(5-1+5)}{2} = \frac{45+2-9}{2} = 19$$

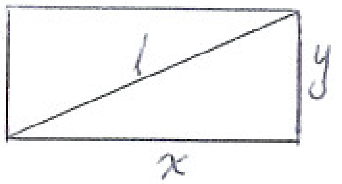
**b)** The **instantaneous velocity** of the particle at $t$=2.

$$= S'(2)$$

$$S'(t) = 10t-1 \Rightarrow S'(2) = 19$$

**Q.11.** Let $l$ be the **length of the diagonal** of a Rectangle. Suppose that **Sides of the Rectangle** have lengths $x$ and $y$. Assume that $x$ and $y$ are **changing with time** $t$.

**a) Draw** a Picture of the Rectangle **with the labels** $x$, $y$ and $l$.




**b)** How are $x$, $y$ and $l$ **related**?

$$l^2 = x^2+y^2$$

**c)** How are $dx/dt$, $dy/dt$ and $dl/dt$ **related**?

$$2l\frac{dl}{dt} = 2x\frac{dx}{dt}+2y\frac{dy}{dt} \Rightarrow l\frac{dl}{dt} = x\frac{dx}{dt}+y\frac{dy}{dt}$$

**d)** As $x$ **increases** at a constant rate of ½ ft/s and $y$ **decreases** at a constant rate of ¼ ft/s, how fast is the **size of Diagonal** changing when $x = 3$ ft and $y = 4$ ft.

$$\frac{dx}{dt} = \frac{1}{2}, \quad \frac{dy}{dt} = -\frac{1}{4}, \quad \left.\frac{dl}{dt}\right|_{\substack{x=3\\y=4}} = ?$$

From (c), we have

$$\frac{dl}{dt} = \frac{x}{l}\frac{dx}{dt}+\frac{y}{l}\frac{dy}{dt}$$

when $\left.\begin{matrix}x=3\\y=4\end{matrix}\right\} \Rightarrow l = \sqrt{3^2+4^2} = 5$

$$\therefore \left.\frac{dl}{dt}\right|_{\substack{x=3\\y=4}} = \frac{3}{5}\left(\frac{1}{2}\right)+\frac{4}{5}\left(-\frac{1}{4}\right)$$

$$= \frac{3}{10}-\frac{4}{20}$$

$$= \frac{1}{10}$$

**e)** Is the length of Diagonal **Increasing or Decreasing** at that instant? Give Reason.

Increasing, Since $\frac{dl}{dt} > 0$.

**Q.12.** **(a)** Find the **Local Linear Approximation** of $\frac{1}{\sqrt{x+2}}$ **at** $x_0 = 2$.

$f(x) \simeq f(x_0) + f'(x_0)(x - x_0)$

$f(x) = \frac{1}{\sqrt{x+2}} = (x+2)^{-\frac{1}{2}}$ , $x_0 = 2$

$f'(x) = -\frac{1}{2}(x+2)^{-\frac{3}{2}} = -\frac{1}{2(x+2)^{3/2}} = \frac{-1}{2(x+2)\sqrt{x+2}}$

$f'(x_0) = f'(2) = \frac{-1}{2(4)\sqrt{4}} = \frac{-1}{16}$

$\therefore f(x) \simeq f(2) + f'(2)(x-2)$

$= \frac{1}{\sqrt{4}} + \left(-\frac{1}{16}\right)(x-2)$

$= \frac{1}{2} - \frac{x-2}{16} = \frac{8-x+2}{16} = \frac{10-x}{16}$

$\therefore \frac{1}{\sqrt{x+2}} \approx \frac{10-x}{16}$

**(b) Use (a)** to **approximate** $\frac{1}{\sqrt{4.01}}$

$\frac{1}{\sqrt{4.01}} = \frac{1}{\sqrt{2.01+2}} \approx \frac{10-2.01}{16}$

$\therefore \frac{1}{\sqrt{4.01}} \approx \frac{7.99}{16}$

**Q.13.** The **side of a square** is measured with a possible **percentage error** of $\pm 1\%$. Use **differentials** to estimate the **percentage error in the area.**

$x :=$ length of the side

$\frac{dx}{x} = \pm .01$

Area $A = x^2$

$dA = 2x\,dx$

$\frac{dA}{A} = \frac{2x}{x^2}dx = 2\frac{dx}{x}$

$\therefore \frac{dA}{A} = 2(\pm .01) = \pm .02$

The Percentage error in the area is $\pm 2\%$

**Part (III)** **(8+9 pts)**

**Q.14.** Water is pouring **at the rate of** 6ft$^3$/min in a **cylindrical tank** of **radius** 120ft. **How fast** is the **height of water** rising up in the cylinder? [**Note:** Volume of Cylinder = $\pi r^2 h$].
**(Show complete work)**

$V = \pi r^2 h$ , $\frac{dv}{dt} = 6$ , $r = 120$

$\frac{dh}{dt} = ?$

$V = \pi (120)^2 h = 14400 \pi h$

$\frac{dv}{dt} = 14400\pi \frac{dh}{dt}$

$6 = 14400\pi \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{6}{14400\pi}$

$= \frac{1}{2400\pi}$ ft/min

**Q.15. Find value(s)** of $k$ so that $f(x) = \begin{cases} \frac{\sin 2k(x-1)}{7(x-1)}, & x \neq 1 \\ k-1, & x = 1 \end{cases}$ **is continuous at** $x = 1$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{\sin 2k(x-1)}{7(x-1)}$

$= \frac{2k}{7} \lim_{x\to 1} \frac{\sin 2k(x-1)}{2k(x-1)}$

$= \frac{2k}{7} \cdot 1 = \frac{2k}{7}$

For f, to be continuous at $x=1$, we must have:

$\lim_{x\to 1} f(x) = f(1)$

$\Rightarrow \frac{2k}{7} = k-1$

$2k = 7k - 7$

$k = \frac{7}{5}$