Lecture 8

Subdifferentiability

The function $F : V \longrightarrow \overline{I\!R}$ is called subdifferentiable at $u \in V$ if there exists a $u^* \in V^*$ such that $\forall v \in V, < u - v, u^* > +F(u)$ is a caf minorant of F. The set of all subgradients (may be empty) at u is denoted by $\partial F(u)$.

Proposition 1 *$u^* \in \partial F(u)$ iff $F(u) + F^*(u^*) = \langle u, u^* \rangle$.*

Proof. If $u^* \in \partial F(u)$

$$\begin{array}{rcl} \langle v-u, u^* \rangle & \leq & F(v) \\ -\langle u, u^* \rangle + F(u) & \leq & -\langle v, u^* \rangle + F(v) \\ \langle u, u^* \rangle - F(u) & \geq & \langle v, u^* \rangle - F(v) \end{array}$$

Taking the supremum over all $v \in V$, we get

$$F^*(u^*) \geq \langle u, u^* \rangle - F(u) \geq \sup_{v \in V} \langle v, u^* \rangle - F(v) \geq F^*(u^*)$$

This shows that $F(u) + F^*(u^*) = \langle u, u^* \rangle$.
Now if $F(u) + F^*(u^*) = \langle u, u^* \rangle$, then $F(u) + \langle v, u^* \rangle - F(v) \leq F(u) + F^*(u^*) = \langle u, u^* \rangle$. Hence

$$F(v) \geq \langle v-u, u^* \rangle + F(u)$$

Which implies that $u^* \in \partial F(u)$. ■

Proposition 2 *$u^* \in \partial F(u)$, then $F^{**}(u) = F(u)$ and $u^* \in \partial F^{**}(u)$.*

Proof. If $u^* \in \partial F(u)$, then $\langle v-u, u^* \rangle + F(u) \leq F^{**}(u) \leq F(v)$ (because from previous proposition, we have $F(u) + F^*(u^*) = \langle u, u^* \rangle$ also $\langle v-u, u^* \rangle + F(u) \leq F(v)$. Now

$$\langle v-u, u^* \rangle + F(u) = \langle v, u^* \rangle - \langle u, u^* \rangle + F(u) = \langle v, u^* \rangle + F^*(u^*) \leq F^{**}(u)$$

So $\langle v-u, u^* \rangle + F(u) \leq F^{**}(u) \leq F(v)$)

From this we conclude that $u^* \in \partial F^{**}(u)$. Furthermore, at $v = u$, we have

$$F(u) \leq F^{**}(u) \leq F(u) \Rightarrow F^{**}(u) = F(u)$$

Now if $F^{**}(u) = F(u)$, then $\partial F(u) = \partial F^{**}(u)$.

$$\begin{array}{rcl} \partial F(u) & = & \{u^* \in V^* : F(u) + F^*(u) =< u, u^* >\} \\ & = & \{u^* \in V^* : F(u) + F^*(u) \leq< u, u^* >\} \\ & = & \{u^* \in V^* : F^*(u) - < u, u^* > \leq F(u)\} \end{array}$$

Since $F^* \in \Gamma(V^*)$ (hence F^* is lsc and convex), then $\partial F(u)$ is closed and convex and $\sigma(V^*, V)$ closed. ■

Theorem 3 *If $F : V \longrightarrow \overline{I\!R}$ is convex, continuous and finite at $u \in V$, then $\partial F(v) \neq \phi$ for all $v \in \overbrace{\mathrm{dom} F}^{\circ}$. In particular $\partial F(u) \neq \phi$.*

Proof.

1. $\overbrace{\mathrm{dom} F}^{\circ} \neq \phi$, F is continuous on $\overbrace{\mathrm{dom} F}^{\circ}$ and is proper over V.

2. $\overbrace{\mathrm{epi} F}^{\circ} \neq \phi$ ($u \in \overbrace{\mathrm{dom} F}^{\circ}$, F is bounded in a neighbourhood $\mathcal{O}_u$ i.e. $F(v) \leq m$ for all $v \in \mathcal{O}_u$ that means $\mathcal{O}_u \times (m + \epsilon, \infty) \in \mathrm{epi} F$).

3. The set of points $(u, F(u)) \forall\, u \text{dom} F$ are boundary points of epi$F$. epi$F = \overset{\circ}{\overbrace{\text{epi}F}}$ +bd(epi$F$).
4. $(u, F(u))$ is a support point for epi$F$ for each $u \in \text{dom}F$.
5. Let $v \in \overbrace{\text{dom}F}$. Since $(v, F(v))$ is a support point for epi$F$, then there is a hyperplane H:

$$\langle w, u^* \rangle + \alpha a + \beta = 0$$

such that $(v, F(v)) \in H$ and $\langle w, u^* \rangle + \alpha a + \beta \geq 0$ for all $(w, a) \in \text{epi}F$.

$$(v, F(v) \in H \Rightarrow \beta = -\langle v, u^* \rangle - \alpha F(v)$$

So H is

$$\langle w - v, u^* \rangle + \alpha(a - F(v)) = 0$$

α must be positive; take $\bar{a}$ sufficiently large then $(v, \bar{a}) \in \overset{\circ}{\overbrace{\text{epi}F}}$ So

$$\alpha(\bar{a} - F(v)) \geq 0 \Rightarrow \alpha \geq 0$$

Assume that $\alpha = 0$. Then $\langle w, u^* \rangle + \beta = 0$ for all $\langle w, a \rangle \in H$.

■