Lecture 22 (Carathéodory Mappings)

DEFINITION 1 (CARATHÉODORY MAPPINGS)
Let $\Omega \in I\!R^m$ be an open Borel set [1], E and F Banach spaces, $g : \Omega \times E \longrightarrow F$. g is called a Carathéodory mapping if

1. $g(\cdot, \zeta)$ is measurable for each $\zeta \in E$.
2. $g(x, \cdot)$ is continuous for almost all $x \in \Omega$.

Let $\mathcal{M}(\Omega, E)$ be the set of measurable functions $u : \Omega \longrightarrow E$, $\mathfrak{m}(\Omega, F)$ the set of measurable functions $v : \Omega \longrightarrow F$. Define $K : (\Omega, F) \longrightarrow \mathfrak{m}(\Omega, F)$ by

$$(Ku)(x) = g(x, u(x))), \qquad x \in \Omega$$

PROPOSITION 2
If $K : L^p(\Omega, E) \longrightarrow L^r(\Omega, F)$. Then K is continuous [2]
with respect to the norms of $L^p(\Omega, E), L^r(\Omega, F)$.

For $E = I\!R^m, F = R, u = \Omega \longrightarrow I\!R^m[u(x) = (u_1(x), u_2(x), \cdots, u_n(x))]$, assume $u \in L^{\alpha_1} \times L^{\alpha_2} \times \cdots \times L^{\alpha_n} = V$. Also assume $Ku(x) = g(x, u(x))$ maps V into $L'(\Omega)$. We can then define $G : V \longrightarrow I\!R$ by

$$G(u) = \int_\Omega Ku(x)dx = \int_\Omega g(x, u(x))dx$$

The conjugate function $G^* : V^* \longrightarrow R$ where

$$V^* = L^{\alpha'_1} \times L^{\alpha'_2} \times \cdots \times L^{\alpha'_n}$$

where $\frac{1}{\alpha_i} + \frac{1}{\alpha'_i} = 1$ for all i is given through the following proposition.

PROPOSITION 3

$$G^*(u^*) = \int_\Omega g^*(x, u^*(x))dx$$

where

$$g^*(x, y) = \sup_{\eta \in I\!R^m} \eta \cdot y - g(x, u)$$

First Examples
$\Omega \subseteq I\!R$ open, given $f \in L^2(\Omega)$,

$$\begin{aligned} -\Delta u &= f \\ u &= 0 \quad \text{on } \Gamma \end{aligned}$$

Variational Form
$V = H_0^1(\Omega)$, let $v \in V$

$$\int_\Omega -\Delta uvdx = \int_\Omega fudx$$

$\langle \nabla u, \nabla v \rangle = \langle f, u \rangle$ for all $v \in V$. This is equivalent to

$$\min \frac{1}{2}||\nabla u||^2 - \langle f, u \rangle$$

[1] For any topological space X, the Borel sigma algebra of X is the σ–algebra $\mathcal{B}$ generated by the open sets of X. In other words, the Borel sigma algebra is equal to the intersection of all sigma algebras $\mathcal{A}$ of X having the property that every open set of X is an element of $\mathcal{A}$. An element of $\mathcal{B}$ is called a Borel subset of X, or a Borel set.

[2] Given $\epsilon > 0, \exists \delta > 0$ such that for all $u, v \in L^p(\Omega, E)$ we have

$$||u - v||_{L^p(\Omega,E)} \leq \delta \Rightarrow ||Ku - Kv||_{L^r(\Omega,F)} \leq \epsilon$$

That is

$$\left(\int_\Omega ||u(x) - v(x)||_E^p dx\right)^p \leq \delta \Rightarrow \left(\int_\Omega ||Ku(x) - Kv(x)||_F^r dx\right)^r \leq \epsilon$$

Side Notes:

- Green's Form
$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f u dx$$
- $\langle u, v\rangle + \sum \langle p_i u, p_i v\rangle = \langle u, v\rangle + \langle \nabla u, \nabla v\rangle$.
- To find the Gâteaux derivative of $F(u)$, we evaluate
$$\frac{d}{dt}F(u+tv)\Big|_{t=0}$$

So
$$\frac{1}{2}||\nabla(u,tv)||^2 - \langle f, u+tv\rangle = \frac{1}{2}||\nabla u||^2 + t\langle u, \nabla v\rangle + \frac{1}{2}t^2||\nabla v||^2 - \langle f, u\rangle - \langle f, tv\rangle$$
Differentiating
$$\langle \nabla u, \nabla v\rangle + t||\nabla u||^2 - \langle f, v\rangle\big|_{t=0} = \langle \nabla u, \nabla v\rangle - \langle f, v\rangle = \min J(u)$$
where
$$J(u) = -\langle f, u\rangle + \frac{1}{2}||\nabla u||^2 = F(u) + G(Au)$$
That is
$$F(u) = -\langle f, u\rangle, \qquad Au = \nabla u, \qquad G(p) = \frac{1}{2}||p||^2$$
Now, we have $V = H_0^1(\Omega), Y = [L^2(\Omega)]^n = Y^*, A : V \longrightarrow Y$ and $V^* = H^{-1}(\Omega)$ (just the dual space of V). Also
$$\phi(u,p) = F(u) + G(Au - p)$$
which belongs to $\Gamma_0(V \times Y)$; since F is convex and G is convex and continuous. We now find the dual problem; so we need to find first F^*.
$$F^*(u^*) = \sup_{u\in V}\langle u, u^*\rangle + \langle f, u\rangle = \sup_{u\in V}\langle u, u^* + f\rangle = \begin{cases} 0, & \text{if } u + f = 0 \\ +\infty & \text{otherwise}\end{cases}.$$
Then G^*. Since $G(p) = \frac{1}{2}\int_\Omega ||p(x)||^2 dx$, we have
$$G^*(p^*) = \int_\Omega \left(\frac{1}{2}|p(x)|^2\right)^* dx$$
To find $\left(\frac{1}{2}|p(x)|^2\right)^*$ let us define $g : \Omega \times I\!R^n \longrightarrow I\!R$ by
$$g(x,y) = \frac{1}{2}||y||^2$$
Then
$$g^*(x,y) = \sup_{\eta\in I\!R^n} \eta y - \frac{1}{2}|y|^2$$
To find the supremum, we shall find the derivative, then equate with zero. Let $\tilde{F}(\eta) = \eta y - \frac{1}{2}|\eta|^2$, then
$$\tilde{F}(\eta + t\zeta) = (\eta + t\zeta)\cdot y - \frac{1}{2}|\eta + t\zeta|^2 = \eta y + t\zeta y - \frac{1}{2}(|\eta|^2 + 2t\eta\zeta + t^2|\zeta|^2)$$
Therefore,
$$\begin{aligned} \frac{d}{dt}\tilde{F}(\eta + t\zeta)\Big|_{t=0} &= 0 \\ \eta y - \eta\zeta - t|\zeta|^2\big|_{t=0} &= \zeta y - \zeta\eta = \zeta(y - \eta) = 0, \quad \forall\, \zeta \in I\!R^n \end{aligned}$$

So for $\eta = y$ we get

$$g^*(x,y) = |y|^2 - \frac{1}{2}|y|^2 = \frac{1}{2}|y|^2$$

$$\therefore G^*(p^*) = \int_\Omega \frac{1}{2}|p^*(x)|^2 dx = \frac{1}{2}||p^*(x)||^2$$

Let us find $A^* : Y^* \longrightarrow V^*$

$$\langle Au, p\rangle = \langle \nabla u, p\rangle = \int_\Omega \nabla u \cdot p dx \overset{\text{Green's}}{=} -\int_\Omega u \nabla p \, dx = \langle u, A^* p\rangle$$

So,

$$A^* p = -\nabla \cdot p$$

Summary:

$F(u)$	$=$	$-\langle f, u\rangle$
$F^*(u^*)$	$=$	$\begin{cases} 0 & u^* = -f \\ +\infty & \text{otherwise} \end{cases}$
$G(p)$	$=$	$\frac{1}{2}\|\|p\|\|^2$
$G^*(p^*)$	$=$	$\frac{1}{2}\|\|p^*\|\|^2$
$A(u)$	$=$	∇u
$A^*(p)$	$=$	$\nabla \cdot p$