Lecture 12

Assumptions : V is a reflexive Banach space, $\varnothing \neq C \subseteq V$ is closed and convex, $F : C \longrightarrow \mathbb{R}$ convex and lower simicontinuous.

Result: under the above assumptions if C is bounded or F is coercive, $a(u,u)$ is a bilinear continuous form satisfiying

$a(u,u) \geq \gamma \|u\|^2, \gamma > 0, l \in V^*$, then $F(u) = a(u,u) - 2\langle l, u\rangle$ has a unique minimazer.

Proposition1: If $F : \varnothing \neq C \longrightarrow \mathbb{R}$ covex, F' exists on C, $u \in C$. TFAE

(i) u minimize F on C

(ii) $\langle F'(u), v-u\rangle \geq 0$ for all $u \in C$

(iii) $\langle F'(v), v-u\rangle \geq 0$ for all $v \in C$

Finding the derivitive of $F(u) = a(u,u) - 2\langle l, u\rangle$,indeed;

$$\begin{aligned}
\lim_{\lambda \longrightarrow 0^+} \frac{F(u+\lambda v) - F(u)}{\lambda} &= \lim_{\lambda \longrightarrow 0^+} \frac{a(u+\lambda v, u+\lambda v) - 2\langle l, u+\lambda v\rangle - a(u,u) + 2\langle l, u\rangle}{\lambda} \\
&= \lim_{\lambda \longrightarrow 0^+} \frac{a(u,u) + \lambda a(u,v) + \lambda a(v,u) + \lambda^2 a(v,v) - 2\langle l,u\rangle - 2\lambda\langle l,v\rangle - a(u,u) + 2\langle l,u\rangle}{\lambda} \\
&= \lim_{\lambda \longrightarrow 0^+} \frac{\lambda a(u,v) + \lambda a(v,u) + \lambda^2 a(v,v) - 2\lambda\langle l,v\rangle}{\lambda}
\end{aligned}$$

$\Rightarrow \langle F'(u), v\rangle = a(v,u) + a(u,v) - 2\langle l, v\rangle$

Remark: if $a(u,u)$ is symmetric, then $\langle F'(u), v\rangle = 2a(u,v) - 2\langle l, v\rangle$

Characterization of the minimizer

$u \in C$ minimizes F iff

(i) $a(u, v-u) - \langle l, v-u\rangle \geq 0$

(ii)$a(v, v-u) - \langle l, v-u\rangle \geq 0$

proposition 2: let $F_1, F_2 : C \longrightarrow \mathbb{R}$ be convex functions,C convex, F_1' exists, $u \in C$,TFAE

(i) u minimizes $F = F_1 + F_2$

(ii) $\langle F_1'(u), v-u\rangle + F_2(v) - F_2(u) \geq 0$ for all $v \in C$

(iii) $\langle F_1'(v), v-u\rangle + F_2(v) - F_2(u) \geq 0$ for all $v \in C$

proof

(i)$\Longrightarrow$(ii)

$0 \leq \frac{F_1((1-\lambda)u+\lambda v) - F_1(u)}{\lambda} + \frac{F_2((1-\lambda)u+\lambda v) - F_2(u)}{\lambda} \leq \frac{F_1(u+\lambda(v-u)) - F_1(u)}{\lambda} + \frac{(1-\lambda)F_2(u)+\lambda F_2(v) - F_2(u)}{\lambda} \leq \frac{F_1(u+\lambda(v-u)) - F_1(u)}{\lambda} + F_2(v) - F_2(u)$ by taking the limit as $\lambda \longrightarrow 0^+$ we get (ii)

(ii) $\Longrightarrow$ (iii) using the convexity of F_1

$F_1(v) \geq F_1(u) + \langle F_1'(u), v-u\rangle$

$F_1(u) \geq F_1(v) + \langle F_1'(v), u-v\rangle$ by adding these two inequlities we obtain

$0 \geq \langle F_1'(u), v-u\rangle + \langle F_1'(v), u-v\rangle \Longrightarrow \langle F_1'(v), v-u\rangle \geq \langle F_1'(u), v-u\rangle \Longrightarrow \langle F_1'(v), v-u\rangle + F_2(v) - F_2(u) \geq \langle F_1'(u), v-u\rangle + F_2(v) - F_2(u) \geq 0$

(iii)$\Longrightarrow$(i)

since C is convex $\Longrightarrow \lambda u + (1-\lambda)v \in C$, $\lambda \in (0,1)$ using (iii) we have

$\langle F_1'(\lambda u + (1-\lambda)v), (1-\lambda)(v-u)\rangle + F_2(\lambda u + (1-\lambda)v) - F_2(u) \geq 0 \Longrightarrow$(by using the convexity of F_2)

$(1-\lambda)\langle F_1'(\lambda u + (1-\lambda)v), (v-u)\rangle + (1-\lambda)(F_2(v) - F_2(u)) \geq 0$ (dividinig by $(1-\lambda)$) we have

$\langle F_1'(\lambda u + (1-\lambda)v), (v-u)\rangle + F_2(v) - F_2(u) \geq 0 \Longrightarrow \langle F_1'(\lambda u + (1-\lambda)v), (v-u)\rangle \geq F_2(u) - F_2(v)$ but

$F_1(v) \geq F_1(\lambda u + (1-\lambda)v) + \langle F_1'(\lambda u + (1-\lambda)v), \lambda(v-u)\rangle \geq F_1(\lambda u + (1-\lambda)v) + \lambda(F_2(u) - F_2(v)) \Longrightarrow$

$F_1(v) + \lambda F_2(v) \geq F_1(\lambda u + (1-\lambda)v) + \lambda F_2(u)$ (by letting $\lambda \longrightarrow 1^-$) we get $F_1(v) + F_2(v) \geq F_1(u) + F_2(u) \Longrightarrow F(v) \geq F(u)$

which completes the proof.

Example1: Proximity Mapping

Let V be a Hilbert space, $x \in V, \varphi \in \Gamma_0(V)$. Define $F(u) = \frac{1}{2}\|u-x\|^2 + \varphi(u)$. set $F_1(u) = \frac{1}{2}\|u-x\|^2$ and $F_2(u) = \varphi(u)$

i) F is strictly convex since F_1 is strictly convex.

ii) F is coercive, indeed; since $\varphi \in \Gamma_0(V)$, there exists a $l \in V^*$, $\alpha \in \mathbb{R}$ such that $\varphi(u) \geq \langle l, u\rangle + \alpha \Longrightarrow$ $F(u) \geq \frac{1}{2}\|u - x\|^2 + \langle l, u\rangle + \alpha \Longrightarrow$

$F(u) \geq \frac{1}{2}(\|u\| - \|x\|)^2 - \|l\|\,\|u\| - |\alpha| \Longrightarrow F(u) \longrightarrow \infty$ as $u \longrightarrow \infty$. hence F is coercive. By proposition 1 F has a unique minimazer.

Evaluating the derevitive of $F_1(u)$.

$\acute{F_1}(u) = \lim\limits_{\lambda \longrightarrow 0^+} \frac{F_1(u+\lambda v) - F_1(u)}{\lambda} = \lim\limits_{\lambda \longrightarrow 0^+} \frac{\frac{1}{2}\|u+\lambda v - x\|^2 - \frac{1}{2}\|u-x\|^2}{\lambda} = \lim\limits_{\lambda \longrightarrow 0^+} \frac{\frac{1}{2}\|u-x\|^2 + \lambda\langle u-x, v\rangle + \frac{1}{2}\lambda^2\|v\| - \frac{1}{2}\|u-x\|^2}{\lambda} =$ $\langle u - x, v\rangle$

by using proposition 2 : u is a minimizer if and only if

i) $\langle u - x, v - u\rangle + \varphi(v) - \varphi(u) \geq 0$ and ii) $\langle v - x, v - u\rangle + \varphi(v) - \varphi(u) \geq 0$.

Special case: if C is a non empty closed convex subset of V, $x \in v$

Define $F(u) = \frac{1}{2}\|u - x\|^2 \Longrightarrow \widetilde{F}(u) = \frac{1}{2}\|u - x\|^2 + \chi_c(u)$ by using the above arqument we have

$\langle u - x, v - u\rangle + \chi_c(v) - \chi_c(u) \geq 0$ and $\langle v - x, v - u\rangle + \chi_c(v) - \chi_c(u) \geq 0 \Longrightarrow$

$\langle u - x, v - u\rangle \geq 0$ for all $v \in C$ and $\langle v - x, v - u\rangle \geq 0$ for all $v \in C$. the mapping $x \longrightarrow u$ is called aproximity mapping and we write

$u-$prox x.