# 1 Convex Sets and their separation

Let $V$ be a vector space, $u, v \in V$. Then

- The line segment between $u$ and $v$ is $[u, v] = \{\lambda u + (1-\lambda)v : \lambda \in [0,1]\}$.
- $A \subseteq V$ is convex iff $A = \{\sum_{i=1}^n \lambda_i u_i : \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, u_i \in A\}$.
- $\phi$ is convex.
- $A \subseteq V$, $co(A) =$ convex hull of $A =$ smallest convex set containing $A = \{\sum_{i=1}^n \lambda_i u_i : \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, u_i \in A\}$
- $V'$ denotes the set of linear functional on $V$.
- A hyperplane $H \subseteq V$ is defined by $H = \{u \in V : l(u) = \alpha \text{ for some } l \in V', \alpha \in I\!R\}$. If $l(u) = \alpha$ is replaced by $l(u) < \alpha$ or $l(u) > \alpha$ ($l(u) \geq \alpha$ or $l(u) \leq \alpha$), then we have open (closed) half spaces.

## 1.1 Separation of convex sets

Let $V$ be a topological vector space (tvs) over the reals, $u, v \in V, \alpha \in I\!R$. Here, we have $(u, v) \longrightarrow u + v, (u, \alpha) \longrightarrow \alpha v$ are continuous. $V$ is called locally convex space (lcs) [1] if it has a fundamental sysytem of neighborhoods of zero consisiting of convex sets.

- If $A \subseteq V$ is convex, then so are $\overset{\circ}{A}, \overline{A}$.
- If $u \in \overset{\circ}{A}, v \in \overline{A}$, then $[u, v] \subseteq \overset{\circ}{A}$ and $\overline{\overset{\circ}{A}} = \overline{A}$.
- **Definition 1** *(Internal Points) $A$ is convex, a point $u \in A$ is called an internal point of $A$ if every line passing through $u$ intersects $A$ in two distinct points $u_1$ and $u_2$ such that $u \in (u_1, u_2)$.*
  - Every interior point is internal.
  - If $\overset{\circ}{A} \neq \phi$, then every internal point to $A$ is interior.
- $A \subseteq V, \overline{co}(A) =$ closed convex hull of $A =$ intersection of all closed convex sets containing $A$.
- In a locally convex space (lcs), a hyperplane $H$ is closed iff its representing functional is continuous.

**Definition 2** *(Separation of sets by hyperplanes) $A, B \subseteq V$. A hyperplane $H$ is said to (strictly) separates $A$ and $B$ if each one of them is contained in one of the (open) half spaces determined by $H$.*

**Theorem 1** *(Hahm-Banach theorem) $V$ is a vs, $M$ is an affine set of $V$, $\phi \neq A \subseteq V$ convex , there exits a hyperplane $H$ such that $M \subseteq H$ and $A \cap H \neq \phi$.*

- **Corollary 1** *If $\phi \neq A \subseteq V$ is open and convex, $\phi \neq B \subseteq V$ is convex. Then there exists a hyperplane that separates $A$ and $B$.*
- **Corollary 2** *$C$(convex) $, B \subseteq V$ (lcs), $C \cap B = \phi, C \neq \phi B \neq \phi$ and $B$ is compact. Then there exists a hyperplane $H$ which strictly separates $A$ and $B$.*

  **Definition 3** *(Supporting hyperplanes) $A \subseteq V, u \in A$. If there exists $H$ such that $A$ lies on one side of $H$ and $u \in H$, then $u$ is called a supporting hyperplane of $A$ at $u$ and $u$ is called the supporting point.*

- **Corollary 3** *If $A \subseteq V$ (tvs), $\overset{\circ}{A} \neq \phi$ is convex. Then every point in the boundary of $A$ is a supporting point.*
- **Corollary 4** *If $V$ (lcs), $M \subseteq V$ is closed and convex. Then $M$ is the intersection of all closed hyperlanes containing it. boundary of $A$ is a supporting point.*

$\sigma(V, V')$ is called the weakest topology. $V$ is a $T_2$ locally convex space in this topology. $\sigma(V, V')$ is the weakest topology in which $V$ is $T_2$ locally convex. In a locally convex space, every closed convex set is also weakly closed.

[1] A normed space is lcs