Lecture 9

We have seen in the previous lecture the if $F : V \longrightarrow \bar{I\!R}$ is convex, finite ($u \in V, |F(u)| < \infty$) and continuous at u. Then $\partial F(v) \neq \phi$ for all $v \in \overbrace{\text{dom}}^{\circ}$. The following inequality is satisfied for each $u^* \in \partial F(u)$

$$\langle v - u, u^* \rangle + \alpha(a - F(u)) \geq 0, \qquad \forall \quad (v, a) \in \text{epi}F$$

So for $(v, F(v))$ we have

$$\langle v - u, u^* \rangle + \alpha(F(v) - F(u)) \geq 0$$
$$F(v) \geq \langle v - u, -\frac{1}{\alpha}u^* \rangle + F(u)$$

So $-\frac{1}{\alpha}u^* \in \partial F(u)$; which shows that $\partial F(u) \neq \phi$

Relation with Gâteaux derivative

$F : V \longrightarrow \bar{I\!R}, u \in V$. If there exists $u^* \in V^*$ such that

$$F'(u, v) = \lim_{\lambda \to 0+} \frac{F(u + \lambda v) - F(u)}{\lambda} = \langle v, u^* \rangle, \qquad \forall v \in V$$

Then u^* is called the Gâteaux derivative of F at u, denoted b $F'(u)$. $F'(u, v)$ is called the directional derivative of F at u in the direction of v. If F is convex, then the above limits always exists; since $\frac{F(u+\lambda v) - F(u)}{\lambda}$ is nondecreasing function of λ (check it).

Proposition 1
Let $F : V \longrightarrow \bar{I\!R}, u \in V$. If $F'(u)$ exists, then $\partial F(u) = \{F'(u)\}$. Conversely, if F is continuous and finite at u and $\partial F(u)$ consists of only one subgradient, then $F'(u)$ exists and $\partial F(u) = \{F'(u)\}$.

Proof. $F'(u)$ exists; that is

$$\langle v, F'(u) \rangle = \lim_{\lambda \to 0+} \frac{F(u - \lambda v) - F(u)}{\lambda} \leq \frac{F(u - \lambda v) - F(u)}{\lambda}, \qquad \forall \quad \lambda \geq 0$$

Let $u + \lambda v = w$, then

$$\langle \frac{w - u}{\lambda}, F'(u) \rangle \leq \frac{F(w) - F(u)}{\lambda}$$
$$\langle w - u, F'(u) \rangle + F(u) \leq F(w)$$
$$\therefore F'(u) \in \partial F(u)$$

Now, suppose $u^* \in \partial F(u)$

$$\langle v - u, u^* \rangle + F(u) \leq F(v), \qquad v \in V$$

Let $\lambda > 0$, put $v = u + \lambda w$. So we have for all $w \in V$ (using the convexity of F)

$$\langle w, u^* \rangle + F(u) \leq \frac{F(u + \lambda w) - F(u)}{\lambda} \leq \frac{F(u + \lambda_0 w) - F(u)}{\lambda_0} \qquad \text{where } \lambda_0 > \lambda$$

This shows that $F'(u)$ exists. Taking the limit as $\lambda \to 0+$ we have

$$\langle w, u^* \rangle \leq \langle w, F'(u) \rangle \qquad \forall \quad w \in V$$

So $u^* = F'(u)$ (since $\langle -w, u^* \rangle \leq \langle -w, F'(u) \rangle \Rightarrow \langle w, u^* \rangle \geq \langle w, F'(u) \rangle$) ■

Lemma 2
Let $F : A \subseteq V \longrightarrow I\!R$, where A is a convex set, F is Gâteaux differentiable on A. Then $A =$ internal A.

Proof. Let $u \in A$. Since $F'(u)$ exists, then

$$\langle v, F'(u) \rangle = \lim_{\lambda \to 0+} \frac{F(u+\lambda v) - F(u)}{\lambda}$$

Hence, for any $v \in V, u + \lambda v \in A$ for sufficiently small λ. So u is an internal to A. ■

Proposition 3

Let $F : A \subseteq V \longrightarrow I\!R$, where A is a convex set, F is Gâteaux differentiable on A. Then the following statements are equivalent.

(i) F (strictly) convex on A.

(ii) $F(v)(>) \geq F(u) + \langle F'(u), v - u \rangle$.

Proof. $(i) \Rightarrow (ii)$ Suppose that F is strictly convex.

$$\langle w, F'(u) \rangle = \lim_{\lambda \to 0+} \frac{F(u+\lambda w) - F(u)}{\lambda} \leq \frac{F(u+\lambda w) - F(u)}{\lambda}, \qquad \forall \quad \lambda > 0$$

Let $u + \lambda w = v$, then

$$\langle \frac{v-u}{\lambda}, F'(u) \rangle \leq \frac{F(v) - F(u)}{\lambda}$$

So,

$$F(v) \geq \langle v - u, F'(u) \rangle + F(u)$$

Since v is an internal point of A (by previous lemma). Then for $v = \alpha v_1 + (1-\alpha)u, \quad \alpha \in (0,1)$ we have

$$\begin{array}{lll} \alpha F(v_1) + (1-\alpha)F(u) & > & F(v) \geq \langle \alpha v_1 + (1-\alpha)u - u, F'(u) \rangle + F(u) \\ \alpha F(v_1) & > & \alpha \langle v_1 - u, F'(u) \rangle + \alpha F(u) \\ F(v_1) & > & \langle v_1 - u, F'(u) \rangle + F(u) \end{array}$$

This proves the first direction. ■