Lecture 28

Duality by the Minimax Theorem

Saddle points of a function: Properties

PROPOSITION 1
If $L : A \times B \to R$,

$$\inf_{u\in A} \sup_{p\in B} L(u,p) \geq \sup_{p\in B} \inf_{u\in A} L(u,p).$$

Proof.

$$L(v,p) \geq \inf_{u\in A} L(u,p) \quad \forall v \in A, \forall p \in B,$$

then

$$\sup_{p\in B} L(v,p) \geq \sup_{p\in B} \inf_{u\in A} L(u,p),$$

then we have

$$\inf_{u\in A} \sup_{p\in B} L(u,p) \geq \sup_{p\in B} \inf_{u\in A} L(u,p).$$

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DEFINITION 2
a point $(\bar{u},\bar{p}) \in A \times B$ is called a saddle point of L on$A \times B$ if

$$L(\bar{u},p) \leq L(\bar{u},\bar{p}) \leq L(u,\bar{p}), \quad \forall u \in A, \forall p \in B.$$

PROPOSITION 3
if $\exists \alpha \in R$ s.t.

$$L(\bar{u},p) \leq \alpha \quad \forall p \in B,$$

and

$$L(u,\bar{p}) \geq \alpha \quad \forall u \in A,$$

then $(\bar{u},\bar{p})$ is a saddle point of L and

$$L(\bar{u},\bar{p}) = \alpha.$$

Proof.

$$L(u,\bar{p}) \geq \alpha \quad \forall u \in A$$

$\Longrightarrow$

$$L(\bar{u},\bar{p}) \geq \alpha$$

and

$$L(\bar{u},p) \leq \alpha \ \ \forall p \in B$$

then

$$L(\bar{u},\bar{p}) \leq \alpha$$

then we have

$$L(\bar{u},\bar{p}) = \alpha$$

■

PROPOSITION 4
1) if $(\bar{u},\bar{p}) \in A \times B$ is a saddle point of L,then

$$L(\bar{u},\bar{p}) = \max_{p\in B} \min_{u\in A} L(u,p) = \min_{u\in A} \max_{p\in B} L(u,p)$$

2)

$$\text{If } \max_{p\in B} \inf_{u\in A} L(u,p) = \min_{u\in A} \sup_{p\in B} L(u,p) = \alpha,$$

then L has a saddle point $(\bar{u},\bar{p}) \in A \times B$ and $L(\bar{u},\bar{p}) = \alpha$.

Proof. 1) suppose $(\bar{u},\bar{p})$ is a saddle point of L,then

$$L(\bar{u},\bar{p}) \leq L(u,\bar{p}) \implies$$

$$L(\bar{u},\bar{p}) = \inf_{u\in A} L(u,\bar{p}) = \min_{u\in A} L(u,\bar{p}) \leq \sup_{p\in B}\min_{u\in A} L(u,p),$$

and

$$L(\bar{u},\bar{p}) \geq L(\bar{u},p) \implies$$

$$L(\bar{u},\bar{p}) = \sup_{p\in B} L(\bar{u},p) = \max_{p\in B} L(\bar{u},p) \geq \inf_{u\in A}\max_{p\in B} L(u,p),$$

since, inf and sup are attained we have:

$$\begin{aligned} \inf_{u\in A}\max_{p\in B} L(u,p) &= \max_{p\in B} L(\bar{u},p) = L(\bar{u},\bar{p}) \\ &= \min_{u\in A} L(u,\bar{p}) = \sup_{p\in B}\min_{u\in A} L(u,p) \end{aligned}$$

$\implies$

$$L(\bar{u},\bar{p}) = \max_{p\in B}\min_{u\in A} L(u,p) = \min_{u\in A}\max_{p\in B} L(u,p).$$

2) Assume

$$\max_{p\in B}\inf_{u\in A} L(u,p) = \min_{u\in A}\sup_{p\in B} L(u,p) = \alpha,$$

we have,

$$\alpha = \inf_{u\in A} L(u,\bar{p}) \leq L(u,\bar{p}), \qquad \forall u \in A$$

and

$$= \sup_{p\in B} L(\bar{u},p) \geq L(\bar{u},p), \qquad \forall p \in B,$$

then L has saddle point. ∎

PROPOSITION 5
the set of saddle points of L on $A \times B$ is of the form $A_o \times B_o$.

Proof. We need to show that if (u_1,p_1) and $(u_2,p_2) \in A \times B$ are saddle points then (u_1,p_2) is saddle point. we know that

$$L(u_1,p_1) = L(u_2,p_2) = \alpha$$

now,

$$\begin{aligned} L(u_1,p_2) &\leq \alpha, \text{ and} \\ L(u_1,p_2) &\geq \alpha, \end{aligned}$$

then we have (u_1,p_2) is saddle point. ∎

Assumptions on L :

Assume V, Z are reflexive Banach spaces,and

$$A \subseteq V \qquad \text{is closed, convex and non empty,}$$

$$B \subseteq Z \qquad \text{is closed, convex and non empty,}$$

the function L satisfies:

for each $u \in A, L(u,.)$ is concave, u.s.c. on B,

for each $p \in B, L(.,p)$ is convex, l.s.c. on A.

PROPOSITION 6
Under the above assumptions, the set $A_o \times B_o$ is convex.if $L(u,.)$ is strictly concave, then B_o contains at most one element. if $L(.,p)$ is strictly convex, then A_o contains at most one element.

Proof. Assume $A_o \times B_o \neq \Phi$,and let $(u_1, p_1), (u_2, p_2) \in A_o \times B_o, \lambda \in [0, 1]$.

$$\begin{aligned} & L(\lambda(u_1, p_1) + (1-\lambda)(u_2, p_2)) \\ = \; & L(\lambda u_1 + (1-\lambda)u_2, \lambda p_1 + (1-\lambda)p_2) \\ \leq \; & \lambda L(u_1, \lambda p_1 + (1-\lambda)p_2) + (1-\lambda)L(u_2, \lambda p_1 + (1-\lambda)p_2) \\ \leq \; & \lambda L(u_1, p_1) + (1-\lambda)L(u_2, p_2) = \alpha. \end{aligned}$$

If $L(u, .)$ is strictly concave, let $u \in A_o, p_1, p_2 \in B_o, \lambda \in (0, 1)$,we have

$$\begin{aligned} \alpha &= L(u, \lambda p_1 + (1-\lambda)p_2) > \lambda L(u, p_1) + (1-\lambda)L(u, p_2) \\ &= \lambda\alpha + (1-\lambda)\alpha = \alpha, \end{aligned}$$

which is impossible ($\alpha > \alpha$). Similarly If $L(., p)$ is strictly convex, let $u_1, u_2 \in A_o, p_1 \in B_o, \lambda \in (0, 1)$,we have

$$\alpha = L(u_1 + (1-\lambda)u_2, p_1) < \lambda L(u_1, p_1) + (1-\lambda)L(u_2, p_1) = \alpha.$$

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Characterization of a saddle point (differentiable functions)

PROPOSITION 7
Assume $L = l + m$, where

$l(u, .)$ is concave , Gateaux-diff. w.r.t. p,
$l(., p)$ is convex , Gateaux-diff. w.r.t. u,
$m(u, .)$ is concave,
$m(., p)$ is convex,

then $(\bar{u}, \bar{p}) \in A \times B$ is a saddle point of L if and only if

$$\begin{aligned} \left\langle \frac{\partial l}{\partial u}(\bar{u}, \bar{p}), u - \bar{u} \right\rangle + m(u, \bar{p}) - m(\bar{u}, \bar{p}) &\geq 0, \ \forall u \in A, \\ \left\langle \frac{\partial l}{\partial p}(\bar{u}, \bar{p}), p - \bar{p} \right\rangle + m(\bar{u}, p) - m(\bar{u}, \bar{p}) &\leq 0, \ \forall p \in B, \end{aligned}$$

Proof. Assume $(\bar{u}, \bar{p})$ is a saddle point, $\lambda \in (0, 1]$

$$\begin{aligned} & \frac{1}{\lambda}\left[L(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - L(\bar{u}, \bar{p})\right] \\ = \; & \frac{1}{\lambda}\left[l(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - l(\bar{u}, \bar{p}) + m(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - m(\bar{u}, \bar{p})\right] \geq 0, \end{aligned}$$

therefore

$$\begin{aligned} \frac{l(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - l(\bar{u}, \bar{p})}{\lambda} + \frac{m(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - m(\bar{u}, \bar{p})}{\lambda} &\geq 0 \\ \frac{l(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - l(\bar{u}, \bar{p})}{\lambda} + \frac{\lambda m(u, \bar{p}) + (1-\lambda)m(\bar{u}, \bar{p}) - m(\bar{u}, \bar{p})}{\lambda} &\geq 0 \end{aligned}$$

cancelling and taking the limits as $\lambda \longrightarrow 0$, we get

$$\left\langle \frac{\partial l}{\partial u}(\bar{u}, \bar{p}), u - \bar{u} \right\rangle + m(u, \bar{p}) - m(\bar{u}, \bar{p}) \geq 0, \ \forall u \in A,$$

the proof of the second one is analogous.
on the other hand assume the inequalities hold, let $u \in A, \lambda \in (0, 1)$,

$$\begin{aligned} l(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - l(\bar{u}, \bar{p}) &\leq \lambda l(u, \bar{p}) + (1-\lambda)l(\bar{u}, \bar{p}) - l(\bar{u}, \bar{p}) \\ &= \lambda\left[l(u, \bar{p}) - l(\bar{u}, \bar{p})\right]. \end{aligned}$$

now,

$$\begin{aligned} L(u,\bar{p}) - L(\bar{u},\bar{p}) &= l(u,\bar{p}) - l(\bar{u},\bar{p}) + m(u,\bar{p}) - m(\bar{u},\bar{p}) \\ &\geq \frac{l(\bar{u}+\lambda(u-\bar{u}),\bar{p}) - l(\bar{u},\bar{p})}{\lambda} + m(u,\bar{p}) - m(\bar{u},\bar{p}) \geq 0 \end{aligned}$$

then we have

$$L(u,\bar{p}) \geq L(\bar{u},\bar{p}),$$

in the same way, we could prove that

$$L(\bar{u},\bar{p}) \geq L(\bar{u},p),$$

so, $(\bar{u},\bar{p})$ is a saddle point. ■

COROLLARY 8
Assume $L(u,.)$ is concave, cateaux-differentiabe and $L(.,p)$ is convex, cateaux-differentiabe, then $(\bar{u},\bar{p})$ is a saddle point of L on $A \times B$ if and only if

$$\begin{aligned} \left\langle \frac{\partial L}{\partial u}(\bar{u},\bar{p}), u-\bar{u} \right\rangle &\geq 0, \ \forall u \in A, \\ \left\langle \frac{\partial L}{\partial p}(\bar{u},\bar{p}), p-\bar{p} \right\rangle &\leq 0, \ \forall p \in B. \end{aligned}$$

Proof. Let $m = 0$. ■