Lecture 25

Theorem: $-\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ is an isometric isomorphism.
Proof:
we know that, for each $f \in H^{-1}(\Omega)$,

$$\begin{aligned} -\Delta u &= f \\ \gamma_0 u &= 0 \end{aligned}$$

has a unique solution $u \in H_0^1(\Omega)$.
This implies that $-\Delta$ is 1-to-1 and on-to. we need to show it is an isometry, indded;

$$\|-\Delta u\|_{H^{-1}(\Omega)} = \|f\|_{H^{-1}(\Omega)} = \sup_{\substack{v\in H_0^1(\Omega)\\ v\neq 0}} \frac{\langle f, v\rangle}{\|v\|_{H_0^1(\Omega)}} = \sup_{\substack{v\in H_0^1(\Omega)\\ v\neq 0}} \frac{\langle -\Delta u, v\rangle}{\|\nabla v\|} = \sup_{\substack{v\in H_0^1(\Omega)\\ v\neq 0}} \frac{\langle \nabla u, \nabla v\rangle}{\|\nabla v\|} = \|u\|_{H_0^1(\Omega)}$$

* Let $L_0^2(\Omega) = \left\{u \in L^2(\Omega) : \int u = 0\right\}$
note that $L_0^2(\Omega)$ is a Hilbert subspace of $L^2(\Omega)$,indeed;

$$\begin{aligned} \text{let } u_n &\in L_0^2(\Omega) \to u \\ u_n &\to u \Longrightarrow \langle u_n, v\rangle \to \langle u, v\rangle \quad \forall\ v \in L^2(\Omega) \\ &\Longrightarrow \langle u_n, 1\rangle \to \langle u, 1\rangle \Longrightarrow \int u = 0 \end{aligned}$$

Lemma: $\nabla . : H_0^1(\Omega)^n \to L_0^2(\Omega)$ is an isomorphism.

Proof:
1- $R(\nabla .) \subseteq L_0^2(\Omega)$, for $u \in H_0^1(\Omega)$ we need to show $\int \nabla . u = 0$

$$\int_\Omega \nabla . u = \langle \nabla . u, 1\rangle = \langle u, \nabla 1\rangle = 0$$

2- $\nabla .$ is bounded, indeed;

$$\|\nabla . u\|_{L^2}^2 = \left\|\sum \frac{\partial u_i}{\partial x_i}\right\|^2 = \int \left|\sum \frac{\partial u_i}{\partial x_i}\right|^2 \leq n \sum \int \left|\frac{\partial u_i}{\partial x_i}\right|^2 \leq n \sum \int |\nabla u_i|^2 = n\,\|u\|_{H_0^1(\Omega)^n}^2 \Longrightarrow \|\nabla .\| \leq \sqrt{n}$$

3- $\nabla .$ is onto

$$(\nabla .)^* = -\nabla : L_0^2(\Omega) \to H^{-1}(\Omega)^n$$

we show that $-\nabla$ is 1-to-1

$$\begin{aligned} -\nabla u &= 0 \text{ in } H^{-1}(\Omega)^n \Longrightarrow u = c \text{ (constant)} \\ \int c &= 0 \Longrightarrow c \int 1 = 0 \Longrightarrow c = 0 \end{aligned}$$

4- $\nabla .$ is 1-to-1

$$\begin{aligned} \text{let } \nabla . u &= 0 \text{ for some } u \in H_0^1(\Omega)^n \Longrightarrow \langle \nabla . u, v\rangle = 0 \quad \forall\ v \in L_0^2(\Omega) \\ &\quad \text{since } \nabla . \text{ is onto} \\ \Longrightarrow &\quad v = \nabla . w \text{ for some } w \in H_0^1(\Omega)^n \Longrightarrow \langle \nabla . u, \nabla . w\rangle = 0 \ \forall\ w \in H_0^1(\Omega)^n \\ \Longrightarrow &\quad \langle u, -\triangle w\rangle = 0 \ \forall\ w \in H_0^1(\Omega)^n \Longrightarrow \langle u, f\rangle = 0 \ \forall\ f \in H^{-1}(\Omega)^n \Longrightarrow u = 0 \end{aligned}$$

Stokes Problem

Let $V = H_0^1(\Omega)^n$, $V^* = H^{-1}(\Omega)^n, Y = L_0^2(\Omega) = Y^*$.
we need to find $u \in H_0^1(\Omega)^n, p \in L_0^2(\Omega)$ such that

$$\begin{cases} -\triangle u + \nabla p = f, & f \in H^{-1}(\Omega)^n \nabla . u = 0 \end{cases}$$

Let $W = \left\{ u \in H_0^1(\Omega)^n : \nabla . u = 0 \right\}$

$$P : \inf_{u \in H_0^1(\Omega)^n} \frac{1}{2} \|u\|^2_{H_0^1(\Omega)^n} - \langle f, u \rangle + \chi_{\{0\}}(\nabla . u)$$

$$F(u) = \frac{1}{2} \|u\|^2_{H_0^1(\Omega)^n} - \langle f, u \rangle$$

$$A : \nabla . : H_0^1(\Omega)^n \to L_0^2(\Omega)$$

$$A^* : -\nabla : L_0^2(\Omega) \to H^{-1}(\Omega)^n$$

$$G(p) = \chi_{\{0\}}(p) = \begin{cases} 0 & \text{if} \quad p = 0 \infty \quad \text{otherwise} \end{cases}$$

$$G^*(u^*) = 0$$

$$F^*(u^*) = \frac{1}{2} \|u^* + f\|^2_{H^{-1}(\Omega)^n}$$

$$P^* : \sup_{p^* \in Y^*} \frac{-1}{2} \|-\nabla p^* + f\|^2_{H^{-1}(\Omega)^n}$$

P has a unique solution and P^* has a unique solution .

Extremality Condition

$$F(\overline{u}) + F^*(A^* \overline{p}^*) = \langle A^* \overline{p}^*, \overline{u} \rangle$$

$$\frac{1}{2} \|\overline{u}\|^2_{H_0^1(\Omega)^n} - \langle f, \overline{u} \rangle + \frac{1}{2} \|-\nabla \overline{p}^* + f\|^2_{H^{-1}(\Omega)^n} = \langle -\nabla \overline{p}^*, \overline{u} \rangle$$

$$\frac{1}{2} \|\overline{u}\|^2_{H_0^1(\Omega)^n} + \frac{1}{2} \|-\nabla \overline{p}^* + f\|^2_{H^{-1}(\Omega)^n} = \langle -\nabla \overline{p}^* + f, \overline{u} \rangle$$

since $-\triangle$ is an isometry $\Longrightarrow$

$$\frac{1}{2} \|-\triangle \overline{u}\|^2_{H^{-1}(\Omega)^n} + \frac{1}{2} \|-\nabla \overline{p}^* + f\|^2_{H^{-1}(\Omega)^n} = \langle -\nabla \overline{p}^* + f, \overline{u} \rangle$$

$$\|-\triangle \overline{u} + \nabla \overline{p}^* - f\|^2_{H^{-1}(\Omega)^n} = 0$$

$$-\triangle \overline{u} + \nabla \overline{p}^* = f$$

The Direct Proof of The Existence of a Solution for P^*

Suppose P_m is a minimizing sequence

$$\|-\nabla p_m + f\|^2_{H^{-1}(\Omega)^n} \longrightarrow \alpha = \inf \|-\nabla p + f\|^2_{H^{-1}(\Omega)^n}$$

$$\Longrightarrow \|-\nabla p_m\|_{H^{-1}(\Omega)^n} \text{ is bounded} \Longrightarrow -\nabla p_m \rightharpoonup F \qquad \text{weak convergence}$$

$$\langle -\nabla p_m, v\rangle \rightharpoonup \langle F, v\rangle$$

$\Longrightarrow$

$$\{\langle -\nabla p_m, v\rangle\}$$

is bunded $\Longrightarrow$

$$\langle -\nabla p_m, v\rangle = \langle p_m, \nabla . v\rangle \quad \text{by Green's Formula}$$

$$\{\langle p_m, \nabla . v\rangle\} \quad \text{is bounded for each} \quad v \in H_0^1(\Omega).$$

$$\Longrightarrow \{\langle p_m, w\rangle\} \quad \text{is bounded for each} \quad w \in L_0^2(\Omega)$$

By the uniform boundedness principle, $\{p_m\}$ is uniformly bounded in $L_0^2(\Omega)$.so,

$$p_m \rightharpoonup p_0 \quad \text{(subsequnce)}$$

$$-\nabla p_m \rightharpoonup -\nabla p_0$$

claim:
$\|-\nabla p_0 + f\|^2_{H^{-1}(\Omega)^n} = \alpha$,indeed;

$$\|-\nabla p_0 + f\|^2_{H^{-1}(\Omega)^n} = \langle -\nabla p_0 + f,\ -\nabla p_0 + f\rangle = \lim_{m\to\infty} \langle -\nabla p_0 + f,\ -\nabla p_0 + f\rangle \leq \lim_{m\to\infty} \|-\nabla p_0 + f\|_{H^{-1}(\Omega)^n} \|-\nabla p_m + f\|_{H^{-1}(\Omega)}$$

$\Longrightarrow$

$$\|-\nabla p_0 + f\|_{H^{-1}(\Omega)^n} \leq \sqrt{\alpha} \Longrightarrow \|-\nabla p_0 + f\|^2_{H^{-1}(\Omega)^n} \leq \alpha \Longrightarrow \|-\nabla p_0 + f\|^2_{H^{-1}(\Omega)^n} = \alpha$$