Lecture 24

The non-linear Dirichlet problem

$$P^* \qquad \inf \tfrac{1}{\alpha}\|u\|^{\alpha} - < f, u >$$

$$u \in V = W_0^{1,\alpha}(\Omega), \quad f \in V^* = W^{-1,\alpha\prime}(\Omega), \quad Y = L^{\alpha}(\Omega)^n, \quad Y^* = L^{\alpha\prime}(\Omega)^n$$

$$F(u) = - < f, u >, \quad G(p) = \tfrac{1}{\alpha}\|p\|^{\alpha} \quad G^*(p^*) = \tfrac{1}{\alpha\prime}\|p^*\|^{\alpha'},$$

Extremality

$$G(A\bar{u}) + G^*(-\bar{p}^*) + < \bar{p}^*, A\bar{u}) >= 0$$

$$\frac{1}{\alpha}\|A\bar{u}\|^{\alpha} + \frac{1}{\alpha\prime}\|\bar{p}^*\|^{\alpha'} + < \bar{p}^*, A\bar{u}) >= 0$$

$$\frac{1}{\alpha}\int_{\Omega}\sum|D_i\bar{u}|^{\alpha} + \frac{1}{\alpha'}\int_{\Omega}\sum|\bar{p}_i^*|^{\alpha'} + \int_{\Omega}\sum\bar{p}_i^* D_i\bar{u} = 0$$

$$\sum\int_{\Omega}\frac{1}{\alpha}|D_i\bar{u}|^{\alpha} + \frac{1}{\alpha'}|\bar{p}_i^*|^{\alpha'} + \bar{p}_i^* D_i\bar{u} = 0,$$

$\Rightarrow$

$$\frac{1}{\alpha}|D_i\bar{u}|^{\alpha} + \frac{1}{\alpha'}|\bar{p}_i^*|^{\alpha'} + \bar{p}_i^* D_i\bar{u} = 0, \qquad i = 1, 2, \ldots.., n$$

then the extremality relation

$$\bar{p}_i^* = -D_i\bar{u}\,|D_i\bar{u}|^{\alpha-2}.$$

now, $A = \nabla, A^* = -div$,

$$A^*p^* = -f$$

$$-\nabla\cdot\bar{p}^* = -f$$

$$\sum D_i\bar{p}_i^* = f$$

$$f = -\sum D_i(D_i\bar{u}\,|D_i\bar{u}|^{\alpha-2}), \qquad \gamma_0\bar{u} = 0$$

The Neumann Problem

$$V = H^1(\Omega), \quad V^* = (H^1(\Omega))^*, \quad Y = L^2(\Omega)^{n+1} = Y^*$$

$$P \qquad \inf_{u\in H^1(\Omega)} \tfrac{1}{2}\left(\|u\|^2 + \|\nabla u\|^2\right) - < f, u >$$

$$F(u) = - < f, u >, \quad Au =< u, \nabla u > \quad , G(p) = \tfrac{1}{2}\|p\|^2,$$

$$F^*(u^*) = \begin{cases} 0 & \text{if } u^* = -f \\ \infty & \text{otherwise,} \end{cases}$$

as before we have,

$$G^*(p^*) = \frac{1}{2}\|p^*\|^2$$

$$P^* \qquad \sup_{A^*p^*=-f} -\tfrac{1}{2}\|p^*\|^2,$$

Extremality

$$G(A\bar{u}) + G^*(-\bar{p}^*) + < \bar{p}^*, A\bar{u}) >= 0$$

$$\frac{1}{2}\|A\bar{u}\|^2 + \frac{1}{2}\|\bar{p}^*\|^2 + < \bar{p}^*, A\bar{u}) >= 0,$$

or

$$\|A\bar{u} + \bar{p}^*\| = 0$$

$$\bar{p}^* = -A\bar{u} = - < \bar{u}, \nabla\bar{u} >$$

$$\bar{p}_1^* = -\bar{u}, \quad \underbrace{\bar{p}_2^* = -\nabla \bar{u}}_{n-\dim.}$$

Now, let $u \in H^1(\Omega), v \in Y$

$$< Au, v >=< (u, \nabla u), (v_1, v_2) >$$
$$=< u, v_1 > + < \nabla u, v_2 >$$
$$=< u, v_1 > + < u, -div\ v_2 > + < \gamma_0 u, \gamma_0 v_2.\upsilon >_\Gamma =< u, A^* v >$$

for $v = \bar{p}^*, A^* p^* = -f$

$$< u, A^* \bar{p}^* >= - < u, \bar{u} > + < u, \Delta \bar{u} > + < \gamma_0 u, \gamma_0 v_2.\upsilon >_\Gamma$$

$$< u, -f >= - < u, \bar{u} > + < u, \Delta \bar{u} > + < \gamma_0 u, \gamma_0 v_2.\upsilon >_\Gamma, \quad \forall u \in H^1(\Omega),$$

in particular, for $u \in H_0^1(\Omega)$

$$< u, -f >=< u, -\bar{u} + \Delta \bar{u} >$$
$$< u, -f + \bar{u} - \Delta \bar{u} >= 0, \quad \forall u \in H_0^1(\Omega)$$

so we have

$$-\Delta \bar{u} + \bar{u} - \Delta \bar{u} = f, \quad \text{in } (H^1(\Omega))^*$$

and for $uu \in H^1(\Omega)$

$$< \gamma_0 u, \gamma_0 (-\nabla \bar{u}.\upsilon >_\Gamma = 0,$$
$$< \gamma_0 u, -\gamma_0 \frac{\partial \bar{u}}{\partial \upsilon} >_\Gamma = 0, \frac{\partial \bar{u}}{\partial \upsilon} = 0 \quad \text{on } \Gamma.$$

The Stokes Problem

$$V = H_0^1(\Omega)^n, \quad V^* = H^{-1}(\Omega)^n, \quad Y = Y^* = L^2(\Omega),$$

Given $f \in V^*$,find $u \in V, p \in L^2(\Omega)$,such that

$$-\Delta u + \nabla p = f$$
$$\nabla.u = 0$$
$$u = 0 \quad \text{on } \Gamma.$$

Let

$$W = \{u \in H_0^1(\Omega)^n : \nabla.u = 0,$$

this is a Hilbert space.
The minimization problem

$$P \qquad \inf_{u \in W} \tfrac{1}{2} \|\nabla u\|^2 - < f, u >$$
$$= \inf_{u \in V} \tfrac{1}{2} \|\nabla u\|^2 - < f, u > + \chi_{\{0\}}(\nabla.u)$$

$$A = div, \quad F(u) = - < f, u > + \tfrac{1}{2} \|\nabla u\|^2$$

$$G(p) = \chi_{\{0\}}(p) = \begin{cases} 0 & \text{if } p = 0 \\ \infty & \text{otherwise} \end{cases}$$

$$G^*(p^*) = \sup_{p \in Y} < p, p^* > - G(p) = 0$$

$$F^*(u) = \sup_{u \in V} < u, u^* > + < f, u > - \frac{1}{2} \|\nabla u\|^2$$

$$= \sup_{u \in V} < u, u^* > + < f, u > - \frac{1}{2} \|u\|^2_{H_0^1(\Omega)^n}$$

$$= \sup_{u \in V} < u, u^* + f > - \frac{1}{2} \|u\|^2_{H_0^1(\Omega)^n} = \|u^* + f\|_{H^{-1}(\Omega)}$$

the problem

$$\begin{aligned}
&\sup_{u\in V} < u, v^* > -\frac{1}{2}\left\|u\right\|^2 \\
&= \sup_{\alpha} \sup_{\|u\|=\alpha} < u, v^* > -\frac{1}{2}\alpha^2 \\
&= \sup_{\alpha} \sup_{\|v\|=1} \alpha < v, v^* > -\frac{1}{2}\alpha^2 \\
&= \sup_{\alpha} \alpha \left\|v^*\right\| - \frac{1}{2}\alpha^2 = \frac{1}{2}\left\|v^*\right\|^2 .
\end{aligned}$$