Lecture 18

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$J(u,p) : V \times Y \to \overline{R}, \quad A \in \mathcal{L}(Y,Y)$

Define $F : V \to \overline{R}$ by $F(u) = J(u, Au)$
p $\underset{u \in V}{inf} F(u)$
P $\underset{u \in V}{inf} J(u, Au)$
Define $\Phi : V \times Y \to \overline{R}$ by $\Phi(u,p) = J(u, Au - p)$

Clearly if J is convex, then Φ is convex.
If $J \in \Gamma_0(V \times Y)$, then $\Phi \in \Gamma_0(V \times Y)$
To show that $\Phi(u,p) = J(u, Au - p)$ is l.s.c. we have
$\underline{\lim}_{(u,p)\to(u_0,p_0)} \Phi(u,p) = \underline{\lim}_{(u,p)\to(u_0,p_0)} J(u, Au-p)$ (note if we put $w = Au - p \Rightarrow w_\circ = Au_\circ - p_\circ$ and as $(u,p) \to (u_\circ, p_\circ)$ we have by continuity of A that $(u,w) - (u_\circ, w_\circ)$). So we get:
$\underline{\lim}_{(u,w)\to(u_0,w_0)} J(u,w) = J(u_0, w_0) = \Phi(u_0, w_0)$

The dual problem:
$\Phi(u^*,p) = \sup_{(u,p)} (\langle u, u^* \rangle + \langle p, p^* \rangle - J(u, Au - p))$ (set $q = Au - p$)
$= \sup_u \sup_q \langle u, u^* \rangle + \langle Au - q, p^* \rangle - J(u,q)$
$= \sup_u \sup_q \langle u, u^* + A^*p^* \rangle + \langle q, -p^* \rangle - J(u,q)$
$= J^*(u^* + A^*p^*, -p^*) \quad \Rightarrow \quad \Phi^*(0,p^*) = J^*(A^*p^*, -p^*)$
So the dual problem can be written as :
P* : $\sup_{P^* \in Y^*} - J^*(A^*p^*, -p^*)$

Stablitiy:

If inf P=h(p) is finit and $J(u_0, \cdot)$ is bounded above in a nbhd of 0, then P is stable, and $inf\ P = sup\ P^*$ and P^* has solutions.
Existence:
If V is a reflexive Banach space, $J(u, Au) \to \infty$ as $\|u\| \to \infty$. Then P has solutions

Extremality:

$\overline{u}$ is a solution of P and $\overline{p}^*$ is a solution of P* iff $J(\overline{u}, A\overline{u}) + J^*(A^*\overline{p}^*, -\overline{p}^*) = 0$ iff $(A^*\overline{p}^*, -\overline{p}^*) \in \partial J(\overline{u}, A\overline{u})$

Note: $F(u) + F^*(u^*) = \langle u, u^* \rangle$ iff $u^* \in \partial F(u)$
$\langle (\overline{u}, A\overline{u}), (A^*\overline{p}^*, -\overline{p}^*) \rangle = \langle \overline{u}, A^*\overline{p}^* \rangle + \langle A\overline{u}, -\overline{p}^* \rangle = 0$

Lagragian of P:
$-L(u,p^*) = \underset{p \in Y}{sup}(\langle p, p^* \rangle - J(u, Au - p)) = \underset{q \in Y}{sup} \langle Au - q, p^* \rangle - J(u,q) = \langle Au, p^* \rangle - \underset{q \in Y}{sup} \langle q, -p^* \rangle - J_u(q) =$
$\langle Au, p^* \rangle + J_u^*(-p^*)$

If $J(u,p) = F(u) + G(p)$
$J^*(u^*,p^*) = F^*(u^*) + G^*(p^*) = J^*(u^*,p^*) = \sup_{(u,p)} (\langle u, u^* \rangle + \langle p, p^* \rangle - J(u,p)) = \sup_u \sup_p (\langle u, u^* \rangle + \langle p, p^* \rangle - F(u) - G(p))$
$= F^*(u^*) + G^*(p^*)$

P: $\inf_{u \in V} F(u) + G(Au)$

P*: $\sup_{p^* \in Y^*} -[F^*(A^*p^*) + G^*(-p^*)]$

Stability:
inf P is finite, $F(u_0) + G(\cdot)$ is bounded in a nbhd of Au_0

Exsistance:
V is reflexive Banach-space
$F(u) + G(Au) \to \infty$ as $\|u\| \to \infty$

Extremality:
$\overline{u}$ is a solution of P and $\overline{p}^*$ is a solution of P* iff

$J(\overline{u}, A\overline{u}) + J^*(A^*\overline{p}^*, -\overline{p}^*) = 0$
$F(\overline{u}) + G(A\overline{u}) + F^*(A^*\overline{p}^*) + G^*(-\overline{p}^*) = 0$
$[F(\overline{u}) + F^*(A^*p^*)] + [G(A\overline{u}) + G^*(-\overline{p}^*)] = 0$
$[F(\overline{u}) + F^*(A^*\overline{p}^*) - \langle \overline{u}, A^*\overline{p}^* \rangle] + [G(A\overline{u}) + G^*(-\overline{p}^*) - \langle A\overline{u}, -\overline{p}^* \rangle] = 0$
$\therefore F(\overline{u}) + F^*(A^*\overline{p}^*) = \langle \overline{u}, A^*\overline{p}^* \rangle$ and $G(A\overline{u}) + G^*(-\overline{p}^*) = \langle A\overline{u}, \overline{p}^* \rangle$ iff $A^*\overline{p}^* \in \partial F(\overline{u})$ and $-\overline{p}^* \in \partial G(A\overline{u})$

Now :
If $Y = \prod_1^m Y_i \qquad Y^* = \prod_1^m Y_i^*$
$p \in Y \quad \to \quad p = (p_1, p_2, ..., p_m)\ , \qquad p_i \in Y_i$
$G(p) = \sum_1^m G_i(p_i) \qquad G_i : Y_i \quad \to \overline{R}$
$A : V \to Y$
$Au = (A_1 u, A_2 u, ..., A_m u)$

The extremality condition takes the form:
$\overline{u}$ is a solution of P, $\overline{p}_i^*$ is a solution of P* iff

$$F(\overline{u}) + F^*(A^*\overline{p}^*) + \sum_1^m G_i(A_j\overline{u}) + \sum_1^m G_i^*(-\overline{p}_i^*) = 0$$

$F(\overline{u}) + F^*(A^*\overline{p}^*) = \langle \overline{u}, A^*\overline{p}^* \rangle,$
$G_i(A_i\overline{u}) + G_i^*(-\overline{p}_i^*) = \langle A_i\overline{u}, -\overline{p}_i^* \rangle, \qquad i = 1, 2, ..., m$

end of lec#18
