II.3 Multiscale edge detection

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1 The continuous wavelet transform in higer dimensions

In order to define the continuous wavelet trasform in higher dimensions we need the space $L^2(\mathbb{R}^n)$ consisting of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|f(x_1,x_2,\ldots,x_n)|^2\,dx_1dx_2\ldots dx_n<\infty.$$

This notation will be abbreviated as

$$\int_{-\infty}^{\infty}|f(\mathbf{x})|^2\,d\mathbf{x}<\infty.$$

$L^2(\mathbb{R}^n)$ is a Hilbert space with the inner product

$$\langle f,g\rangle=\int_{-\infty}^{\infty}f(\mathbf{x})\,\overline{g}(\mathbf{x})\,d\mathbf{x}$$

and induced norm

$$\|f\|^2=\langle f,f\rangle\,.$$

A wavelet in transform in $\mathbb{R}^n$ is a function $\psi\in L^2(\mathbb{R}^n)$ with the properties

(a) $\|\psi\|=1$,

(b) $\int_{-\infty}^{\infty}\psi(\mathbf{x})\,d\mathbf{x}=0$.

As before, the dilated and translated wavelet $\psi_{a,\mathbf{b}}$ is given by

$$\psi_{a,\mathbf{b}}(\mathbf{x}) = \frac{1}{a^{n/2}} \psi\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right), \ (a, \mathbf{b}) \in \mathbb{R}^{+} \times \mathbb{R}^{n}.$$

The continuous wavelet transform of a function $f \in L^2(\mathbb{R}^n)$ then is then defined by

$$W_{\psi} f(a, \mathbf{b}) = \langle f, \psi_{a,\mathbf{b}} \rangle.$$

An exactly similar inversion formula holds in this case also

$$f(\mathbf{x}) = \int_0^{\infty} \int_{-\infty}^{\infty} W_{\psi} f(a, \mathbf{b}) \, \psi_{a,\mathbf{b}}(\mathbf{x}) \, w(a) \, d\mathbf{b} da.$$

In what follows we will need the notion of a line segement joining two points in $\mathbb{R}^n$.

Definition 1 *(line segments)*

Given two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *, the line segment joining* $\mathbf{x}, \mathbf{y}$, *denoted* $[\mathbf{x}, \mathbf{y}]$, *is defined by*

$$[\mathbf{x}, \mathbf{y}] = \{\gamma \mathbf{x} + (1-\gamma) \mathbf{y} : \gamma \in [0, 1]\}.$$

2 Wavelets and edge detection in images

A gray scale image can be regarded as a function f defined on a recangular box $\mathbf{B} = [a, b] \times [c, d] \subset \mathbb{R}^2$ that assigns to each pixel (picture element) $\mathbf{x} \in \mathbf{B}$ a gray scale level $f(\mathbf{x}) \in [0, 1]$ where 0 stands for black and 1 stands for white. Intuitivley, an edge is where an abrupt change in the gray scale levels occur. The rate of change of in a certain unit direction $\mathbf{d} = (d_1, d_2)$ (i.e. $\|\mathbf{d}\| = 1$) is measured by $\nabla f(\mathbf{x}) \cdot \mathbf{d}$ where

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{x_1}(\mathbf{x}) \\ \frac{\partial f}{x_2}(\mathbf{x}) \end{pmatrix}.$$

is the gradient of f at $\mathbf{x}$. Let's assume for the moment that the fucntion f is smooth, in the sense t maximum rate of change of f at the point $\mathbf{x}$ occurs in the direction $\mathbf{n} = \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}$ of the gradient itself, with the maximum rate of change being $\|\nabla f(\mathbf{x})\|$. The direction $\mathbf{n}$ is called the direction of maximum ascent.

Definition 2 *(Definition of an edge)*

Suppose $f:\mathbf{B}\rightarrow[0,1]$ is a "smooth" image. A point $\mathbf{x}\in\mathbf{B}$ is called an edge point if $\|\nabla f(\mathbf{x})\|$ is locally maximum in the direction of maximum ascent. This means that

$$\|\nabla f(\mathbf{x})\| \geq \|\nabla f(\mathbf{y})\| \ \ \forall \mathbf{y}\in[\mathbf{x}-\varepsilon\mathbf{n},\mathbf{x}+\varepsilon\mathbf{n}],$$

where $\mathbf{n}=\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}$ for some $\varepsilon>0$.

In reality, images are not smooth and gradients are not defined. To overcome this difficulty, an image f is smoothed by convolution with a family θ_a of smooth compactly supported functions. That is we replace f by $f*\theta_a$ and the family θ_a is chosen such that $\lim_{a\rightarrow 0^+} f*\theta_a(\mathbf{x})=f(\mathbf{x})$. In this case

$$\nabla(f*\theta_a)=f*\nabla\theta_a.$$

The family θ_a can be "dilations" of a single compactly supported function θ: $\theta_a(\mathbf{x})=\theta\left(\frac{\mathbf{x}}{a}\right)$. Let

$$\psi^1:=\frac{\partial(\mathcal{R}\theta)}{\partial x_1},\quad \psi^2:=\frac{\partial(\mathcal{R}\theta)}{\partial x_2} \tag{1}$$

and observe that $\psi^1,\ \psi^2$are wavelets. Also

$$\begin{aligned} D_a\mathcal{R}\psi^1(\mathbf{x}) &= D_a\frac{\partial\theta}{\partial x_1}(\mathbf{x})=\frac{1}{a}\frac{\partial\theta}{\partial x_1}\left(\frac{\mathbf{x}}{a}\right) \\ &= \frac{\partial}{\partial x_1}\left(\theta\left(\frac{\mathbf{x}}{a}\right)\right)=\frac{\partial\theta_a}{\partial x_1}(\mathbf{x}) \end{aligned}$$

and similarly

$$D_a\mathcal{R}\psi^2(\mathbf{x})=\frac{\partial\theta_a}{\partial x_1}(\mathbf{x}).$$

Therefore,

$$\begin{aligned} \left\langle f,\psi^1_{a,\mathbf{b}}\right\rangle &= f*D_a\mathcal{R}\psi^1(\mathbf{b}) \\ &= f*\frac{\partial\theta_a}{\partial x_1}(\mathbf{b}) \\ &= \frac{\partial}{\partial x_1}f*\theta_a(\mathbf{b}) \end{aligned}$$

and

$$\left\langle f, \psi^2_{a,\mathbf{b}} \right\rangle = \frac{\partial}{\partial x_2} f * \theta_a (\mathbf{b})$$

We immediately recognize $\nabla \ f * \theta_a$ as the two wavelet transforms

$$W^1 f (a, \mathbf{b}) = \left\langle f, \psi^1_{a,\mathbf{b}} \right\rangle , \quad W^2 f (a, \mathbf{b}) = \left\langle f, \psi^2_{a,\mathbf{b}} \right\rangle .$$

Arranging these two wavelet transforms in a vector

$$\mathbf{W} f (a, \mathbf{b}) = \begin{pmatrix} W^1 f (a, \mathbf{b}) \\ W^2 f (a, \mathbf{b}) \end{pmatrix} \tag{2}$$

we have

$$\mathbf{W} f (a, \mathbf{b}) = \nabla f * \theta_a (\mathbf{b}) .$$

Thus a point $\mathbf{b}$ is an edge point for an image f at the scale a if $\|\mathbf{W} f (a, \mathbf{b})\|$ is a local maximum along the direction of maximum ascent or

$$\|\mathbf{W} f (a, \mathbf{b})\| \geq \|\mathbf{W} f (a, \mathbf{b}')\| \quad \forall \mathbf{b}' \in [\mathbf{b} - \varepsilon \mathbf{n}, \mathbf{b} + \varepsilon \mathbf{n}]$$

where $\mathbf{n} = \frac{\mathbf{W} f(a,\mathbf{b})}{\|\mathbf{W} f(a,\mathbf{b})\|}$ for some $\varepsilon > 0$. These points are also called *wavelet transform modulus maxima points.*

Maxima curves Edges in images are found by chaining together wavelet transform modulus maxima points. Since these points lie on level curves, they are alighned with the local tangents at the points of modulus maxima. These tangents are then orthogonal to the direction of maximum ascent $\mathbf{n} = \frac{\mathbf{W} f(a,\mathbf{b})}{\|\mathbf{W} f(a,\mathbf{b})\|}$. There are two such directions

$$\mathbf{t}_1 = \frac{1}{\|\mathbf{W} f (a, \mathbf{b})\|} \begin{pmatrix} W^2 f (a, \mathbf{b}) \\ -W^1 f (a, \mathbf{b}) \end{pmatrix} , \ \mathbf{t}_2 = -\mathbf{t}_1.$$

In discrete calculations, if $\mathbf{b}$ is a local maxima point, we find the adjacent point to $\mathbf{b}$ which is colsest to $\mathbf{t}_1$ (or $\mathbf{t}_2$).

The edge detection algorithm is as follows:

Algorithm 1 *(edge detection algorithm)*

For a given image f and a scale $a > 0$, let θ be a gaussian in $\mathbb{R}^2$ and define the wavelets ψ^1, ψ^2 as in (1)

1. *Compute the vector wavelet transform $\mathbf{W} f (a, \cdot)$ from(2)*
2. *Locate the modulus maxima of $\mathbf{W} f (a, \cdot)$.*
3. *For each $\mathbf{b}$ find the adjacent edge point as the closest to $\mathbf{t}_1$ (or $\mathbf{t}_2$).*
4. *Form an edge by chaining together the adjacent modulus maxima points.*