V Solving Integral Equations

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1 Introduction

We apply the wavelet transform to approximate the solution of integral equations of the form

$$Tf(x) = \int k(x, y) f(y) \, dy. \tag{1}$$

2 The wavelet hierarchy

Suppose we write the 0 level approximation of $Tf(x), k(x, y)$ and $f(y)$ as

$$\begin{aligned}
Tf(x) &\approx \sum_{k=1}^{M} t_k \varphi_{0,k}(x), \\
k(x, y) &\approx \sum_{k=1}^{M} \sum_{l=1}^{M} c_{kl} \varphi_{0,k}(x) \varphi_{0,l}(y), \\
f(y) &\approx \sum_{m=1}^{M} s_m \varphi_{0,m}(y).
\end{aligned}$$

Then, using the orthonormality of $\{\varphi_{0,k}\}$,

$$\begin{aligned}
\int k(x, y) f(y) \, dy &\approx \sum_k \sum_l \sum_m c_{kl} s_m \varphi_{0,k}(x) \int \varphi_{0,l}(y) \varphi_{0,m}(y) \, dy \\
&= \sum_k \left(\sum_l c_{kl} s_l \right) \varphi_{0,k}(x).
\end{aligned}$$

Comparing with the expansion of the left hand side, we get

$$t_k = \left(\sum_l c_{kl} s_l \right)$$

for each k. This last equation may be written in a matrix form as

$$t = \mathcal{C} s,$$

where $\mathcal{C} = (c_{kl})$ is an $M \times M$ matrix. Solving the integral equation (1), or rather approximating its solution amounts to solving the above matrix equation for s. Performing the integral transform amounts to performing the matrix multiplication $\mathcal{C}s$. We will develop a wavelet approach for performing either task.

The first coarser level decompositions of $Tf(x)$, $k(x,y)$ and $f(y)$ are

$$
\begin{aligned}
Tf(x) &\approx \sum_k t_k^1 \varphi_{-1,k}(x) + \sum_k w_k^1 \psi_{-1,k}(x), \\
k(x,y) &\approx \sum_k \sum_l c_{kl}^1 \varphi_{-1,k}(x)\, \varphi_{-1,l}(y) \\
&\quad + \sum_k \sum_l \alpha_{kl}^1 \varphi_{-1,k}(x)\, \psi_{-1,l}(y) \\
&\quad + \sum_k \sum_l \beta_{kl}^1 \psi_{-1,k}(x)\, \varphi_{-1,l}(y) \\
&\quad + \sum_k \sum_l \gamma_{kl}^1 \psi_{-1,k}(x)\, \psi_{-1,l}(y) \\
f(y) &\approx \sum_l s_l^1 \varphi_{-1,l}(y) + \sum_l d_l^1 \psi_{-1,l}(y).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int k(x,y) f(y)\, dy &\approx \sum_k \left(\sum_l c_{kl}^1 s_l^1 \right) \varphi_{-1,k}(x) + \sum_k \left(\sum_l \beta_{kl}^1 s_l^1 \right) \psi_{-1,k}(x) \\
&\quad + \sum_k \left(\sum_l \alpha_{kl}^1 d_l^1 \right) \varphi_{-1,k}(x) + \sum_k \left(\sum_l \gamma_{kl}^1 d_l^1 \right) \psi_{-1,k}(x) \\
&= \sum_k \left(\sum_l c_{kl}^1 s_l^1 + \sum_l \alpha_{kl}^1 d_l^1 \right) \varphi_{-1,k}(x) \\
&\quad + \sum_k \left(\sum_l \beta_{kl}^1 s_l^1 + \sum_l \gamma_{kl}^1 d_l^1 \right) \psi_{-1,k}(x).
\end{aligned}
$$

Comparing with the expansion of Tf we get

$$
\begin{aligned}
t_k^1 &= \sum_l c_{kl}^1 s_l^1 + \sum_l \alpha_{kl}^1 d_l^1, \\
w_k^1 &= \sum_l \beta_{kl}^1 s_l^1 + \sum_l \gamma_{kl}^1 d_l^1.
\end{aligned}
$$

Again, this system can be written in matrix form as

$$
\begin{bmatrix} w^1 \\ t^1 \end{bmatrix} = \begin{bmatrix} -^1 & \mathcal{B}^1 \\ \mathcal{A}^1 & \mathcal{C}^1 \end{bmatrix} \begin{bmatrix} d^1 \\ s^1 \end{bmatrix}.
$$

We proceed by obtaining second level wavelet decompositions. Letting

$$
\begin{aligned}
Tf^1(x) &= \sum_k t_k^1 \varphi_{-1,k}(x), \\
k^1(x,y) &= \sum_k \sum_l c_{kl}^1 \varphi_{-1,k}(x) \varphi_{-1,l}(y), \\
f^1(y) &= \sum_l s_l^1 \varphi_{-1,l}(y)
\end{aligned}
$$

we get the second level approximations and details as

$$
\begin{aligned}
Tf^1(x) &= \sum_k t_k^2 \varphi_{-2,k}(x) + \sum_k w_k^2 \psi_{-2,k}(x), \\
k^1(x,y) &= \sum_k \sum_l c_{kl}^2 \varphi_{-2,k}(x) \varphi_{-2,l}(y) \\
&\quad + \sum_k \sum_l \alpha_{kl}^2 \varphi_{-2,k}(x) \psi_{-2,l}(y) \\
&\quad + \sum_k \sum_l \beta_{kl}^2 \psi_{-2,k}(x) \varphi_{-2,l}(y) \\
&\quad + \sum_k \sum_l \gamma_{kl}^2 \psi_{-2,k}(x) \psi_{-2,l}(y) \\
f^1(y) &= \sum_l s_l^2 \varphi_{-2,l}(y) + \sum_l d_l^2 \psi_{-2,l}(y).
\end{aligned}
$$

Repeating the integration and comparison steps above, we get

$$
\begin{aligned}
t_k^2 &= \sum_l c_{kl}^2 s_l^2 + \sum_l \alpha_{kl}^2 d_l^2, \\
w_k^2 &= \sum_l \beta_{kl}^2 s_l^2 + \sum_l \gamma_{kl}^2 d_l^2.
\end{aligned}
$$

In matrix form, we have

$$
\begin{bmatrix} w^2 \\ t^2 \end{bmatrix} = \begin{bmatrix} -^2 & \mathcal{B}^2 \\ \mathcal{A}^2 & \mathcal{C}^2 \end{bmatrix} \begin{bmatrix} d^2 \\ s^2 \end{bmatrix}.
$$

Since

$$
\begin{aligned}
f(y) &\approx f^1(y) + \sum_l d_l^1 \psi_{-1,l}(y) \\
&= \sum_l s_l^2 \varphi_{-2,l}(y) + \sum_l d_l^2 \psi_{-2,l}(y) + \sum_l d_l^1 \psi_{-1,l}(y)
\end{aligned}
$$

and

$$
\begin{aligned}
Tf(x) &= Tf^1(x) + \sum_k w_k^1 \psi_{-1,k}(x) \\
&= \sum_k t_k^2 \varphi_{-2,k}(x) + \sum_k w_k^2 \psi_{-2,k}(x) + \sum_k w_k^1 \psi_{-1,k}(x),
\end{aligned}
$$

we see that the level 2 decomposition matrix equation can be written in the form

$$\begin{bmatrix} -^{1} & \mathcal{B}^{1} & & \\ \mathcal{A}^{1} & 0 & & \\ & & -^{2} & \mathcal{B}^{2} \\ & & \mathcal{A}^{2} & \mathcal{C}^{2} \end{bmatrix} \begin{bmatrix} d^{1} \\ s^{1} \\ d^{2} \\ s^{2} \end{bmatrix} = \begin{bmatrix} w^{1} \\ * \\ w^{2} \\ t^{2} \end{bmatrix}.$$

After N steps of decomposition we reach the matrix system

$$\begin{bmatrix} -^{1} & \mathcal{B}^{1} & & & & & \\ \mathcal{A}^{1} & 0 & & & & & \\ & & -^{2} & \mathcal{B}^{2} & & & \\ & & \mathcal{A}^{2} & 0 & & & \\ & & & & \ddots & & \\ & & & & & -^{N} & \mathcal{B}^{N} \\ & & & & & \mathcal{A}^{N} & \mathcal{C}^{N} \end{bmatrix} \begin{bmatrix} d^{1} \\ s^{1} \\ d^{2} \\ s^{2} \\ \vdots \\ d^{N} \\ s^{N} \end{bmatrix} = \begin{bmatrix} w^{1} \\ * \\ w^{2} \\ * \\ \vdots \\ w^{N} \\ t^{N} \end{bmatrix}.$$

The sizes of the submatrices involved in the above calculations are progressively smaller and smaller. For example, the matrix $-^{1}$ is $M/2 \times M/2$, the size of $-^{2}$ is $M/4 \times M/4$ ... the size of $-^{N}$ is $M/2^{N} \times M/2^{N}$.. In particular, if $M = 2^{N}$ then $-^{N}$ is a 1×1 matrix.

In many applications, for example, for solving differential equations or performing integral transforms such as the Hilbert transform, the kernel $k(x, y)$ is smooth away from the diagonal. If smooth wavelets are used, the coefficient matrices contain very small values away from the diagonal. Thus, one may use wavelet thresholding to set the small coefficients to zero. The result is a much sparser matrix of coefficients that can be taken advantage of either for fast multiplication or for solving systems of equations. For example, if after thresholding each of the submatrices is a band r matrix with $r << M$, we can show that the matrix of coefficients contains around $3rM - 3r + 1$ nonzero elements. This is to be compared to the original M^{2} nonzero elements. The compression ratio in this case is about $3r/M$.