0.1 Maximum Principle in R^N

Let Ω be an open set of R^N.

Theorem. (Maximum Principle for the Dirichlet problem).

Let $a_{ij} \in L^\infty(\Omega)$ satisfying the ellipticity (coercivity) condition and $f \in L^2(\Omega)$. If $u \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\int_\Omega \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} + u\phi = \int_\Omega f\phi, \quad \forall \phi \in H_0^1(\Omega) \tag{1}$$

then

$$\min\{\inf_\Gamma u,\ \inf_\Omega f\} \leq u(x) \leq \max\{\sup_\Gamma u,\ \sup_\Omega f\}. \tag{2}$$

Proof. Let's use the transaction method of **Stampacchia**. For this, take $G \in C^1(R)$ such that

(i) $|G'(s)| \leq M, \ \forall s \in R$

(ii) G is strictly increasing over $(0, +\infty)$

(iii) $G(s) = 0, \quad \forall s \leq 0$

We will prove the right-hand part of (2). Suppose that

$$K = Max\{\sup_\Gamma u,\ \sup_\Omega f\} < +\infty$$

Otherwise (2) is satisfied.

Set $v = G(u - K)$. We distinguish two cases:
a) $|\Omega| < +\infty$
In this case, $v \in H^1(\Omega)$ and $v(x) = 0, \ \forall x \in \Gamma$,hence
$v \epsilon H_0^1(\Omega)$. Then use it in (1) to obtain

$$\int_\Omega \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} G'(u-k) + \int_\Omega (u-k)G(u-k) = \int_\Omega (f-k)G(u-k) \tag{3}$$

This gives

$$\int_\Omega (u-k)G(u-k) = -\int \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} G'(u-k) + \int_\Omega (f-k)G(u-k)$$

But

$$\int_\Omega (u-k)G(u-k) = \int_{\Omega_+} (u-k)G(u-k) \leq 0 \tag{4}$$

where

$$\Omega_+ = \{x \,\epsilon\, \Omega \ / \ u - k > 0\}$$

By using (4) and the fact that $(u-k)G(u-k) \geq 0$ in Ω_+ then we have

$$0 \leq \int_{\Omega_+} (u-k)G(u-k) \leq 0$$

Thus, the measure $(\Omega_+) = 0 \Rightarrow u - k \leq 0$ a.e. in Ω.

$$u(x) \leq k \text{a.e. in} \Omega.$$

b) $|\Omega| = +\infty$.
In this case, $k \geq 0$ (since $f(x) \leq k$ a.e. in Ω and $f \in L^2(\Omega)\}$. Take $k' < k \geq 0$ and set $v = G(u-k')$. Then $v \in H^1(\Omega)$, also $v \in C(\bar{\Omega})$ with $v = 0$ on Γ. So, $v \in H_0^1(\Omega)$. We then use it in (1) to get (3); hence the result is established $u(x) \leq k^{'}$ a.e. x in Ω.

Since k' is arbitrary $< k$ then $u(x) \leq k$ a.e. in Ω. This complete the proof.

Remark 1. Since $|\Omega| = +\infty$, we need $\int_\Omega G(u-k') < +\infty$. This is certainly true since

$$\int_\Omega G(u-k') = \int_{\Omega'_+} G(u-k'),$$

where $\Omega'_+ = \{x \in \Omega \ / \ u \geq k'\}$. So, by using

$$G(u-k') = |G(u-k') - G(-k')| \leq M|u|$$

we easily arrive at

$$0 \leq k' \int_{\Omega'_+} G(u-k') \leq \int_{\Omega'_+} u \ M|u| = M \int_{\omega_+} u^2 < +\infty.$$

Remark 2. The left-hand side of (2) can be proved by considering $-f$ and $-u$.

Corollary. Let $f \in L^2(\Omega)$ and $u \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfying (1). we have the following:
a) If $u \geq 0$ on Γ and $f \geq 0$ in Ω then $u \geq 0$ in Ω, with

$$\|u\|_{L^\infty(\Omega)} \leq Max\{\|u\|_{L^\infty(\Gamma)}, \|f\|_{L^\infty(\Omega)}\|$$

In particular, we have
b) If $f = 0$ in Ω then $\|u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Gamma)}$
c) If $u = 0$ on Γ then $\|u\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$

Theorem. Let $a_{ij} \in L^\infty(\Omega)$ satisfying the ellipticity (coercivity) condition and $a_k \in L^\infty$, $0 \leq k \leq N$, with $a_0 \geq 0$ in Ω. Let $f \in L^2(\Omega)$ and $u \in H^1(\Omega) \cap C(\bar{\Omega})$ such that

$$\int_\Omega \sum_{i,j} a_{i_j} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \int_\Omega \sum_{k=1}^{N} a_k \frac{\partial u}{\partial x_i} \phi + \int_\Omega a_0 u \phi = \int f\phi, \quad \forall \phi \ \epsilon \ H_0^1(\Omega) \tag{5}$$

Then

$$(u \geq 0 \text{ and } \Gamma) \text{ and} (f \geq 0 \text{ in } \Omega) \Rightarrow (u \geq 0 \text{ in } \Omega) \tag{6}$$

If $a_0 \equiv 0$ and Ω is bounded. Then

$$(f \geq 0 \text{ in } \Omega) \Rightarrow (u \geq \inf_{\Gamma} u \; in \; \Omega) \tag{7}$$

$$(f = 0 \text{ in } \Omega) \Rightarrow (\inf_{\Gamma} u \leq u \leq \sup_{\Gamma} u \text{ in } \Omega) \tag{8}$$

Proof. We only prove the case $a_k \equiv \; 0 \leq k \leq N$. For the general case, we refer to Gilbarg & Trudinger (Elliptic PDE's of second order, Theorem 8.1).

Now, we prove (6), or equivalently

$$(u \leq 0 \text{ o } \Gamma) \text{ and } (f \leq 0 \text{ in } \Omega) \Rightarrow (u \leq 0 \text{ in } \Omega) \tag{9}$$

Let $\phi = G(u)$, where G is defined earlier.
So, (5) gives

$$\int_{\Omega} \sum a_{i_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} G'(u) \leq 0$$

hence

$$\int_{\Omega} |\nabla u|^2 G'(u) \leq 0$$

But G is nondecreasing. So, $\int_{\Omega} |\nabla u|^2 G'(u) = 0$. Therefore $|\nabla u|^2 G'(u) = 0$. Hence $u \leq 0$.

Next, we establish (7). Set $k = \inf_{\Gamma} u < -\infty$;otherwise (7) is valid. Also $w = u - k$ satisfies (5) since $a_0 \equiv 0$ and $w \in H^1(\Omega)$. since Ω is bounded. Applying (6) to obtain $w \geq 0$ that is $u \geq k = \inf_{\Gamma} f \; u$.

Finally (8) follows from (7) and the fact that

$$(f \leq 0 \text{ in } \Omega) \Rightarrow (u \leq \sup_{\Gamma} u \text{ in } \Omega) \tag{10}$$

which is equivalent to (7).
Theorem (Maximum principle for the Neumann problem)

Let $a_{ij} \in L^{\infty}(\Omega)$ satisfying the ellipticity (coercivity) condition and $f \in L^2(\Omega)$. If $u \in H^1(\Omega)$ satisfies

$$\int_{\Omega} \nabla u \cdot \nabla \phi + \int_{\Omega} u\phi = \int_{\Omega} f\phi, \quad \forall \phi \in H^1(\Omega)$$

then

$$\inf_{\omega} f \leq u(x) \leq \sup_{\Omega} f, \; \forall x \text{ a.e in } \Omega$$

Proof. Similar to the case of Dirichlet problem.