Corollary: Given $m \geq 1$ and $1 \leq p < \infty$. Then

1. For $\frac{N}{p} > m$, we have

$$W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \quad \frac{1}{q} = \frac{1}{p} - \frac{m}{N}$$

2. For $\frac{N}{p} = m$, we have

$$W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \quad \forall\, q \in [p, +\infty)$$

3. For $\frac{N}{p} < m$, we have

$$W^{m,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$$

Moreover if

$$m - \frac{N}{p} = k + \theta, \text{ for } k = \left[m - \frac{N}{p}\right] \text{ and } 0 < \theta < 1$$

then$\forall\, u \in W^{m,p}(\mathbb{R}^N)$ we have

$$||D^\alpha u||_{L^\infty} \leq C||u||_{W^{1,p}}, \quad \forall\, \alpha, \quad |\alpha| \leq k$$

and

$$|D^\alpha u(x) - D^\alpha u(y)| \leq C||u||_{W^{1,p}}|u - y|^\theta$$

for almost every x, y in $\mathbb{R}^N$ and all α, $|\alpha| = k$. In particular,

$$W^{m,p}(\mathbb{R}^N) \subset C^k(\mathbb{R}^N)$$

Remark: To prove the above results, we only reiterate the results of the embedding theorems for successive derivatives.
Corollary. For the special case $p = 1$ and $m = N$, we have $W^{N,1}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$.
Proof. Let $u \in C_0^\infty(\mathbb{R}^N)$, so we have

$$u(x_1, x_2, \ldots, x_N) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{xN} \frac{\partial^N u}{\partial x_1 \partial x_2 \ldots \partial x_N}(t_1, t_2, \ldots, t_N)\, dt_1, \ldots dt_N$$

hence

$$||u||_\infty \leq ||u||_{W^{N,1}}$$

For $u \in W^{N,1}(\mathbb{R}^N)$, we use the density of $C_0^\infty(\mathbb{R}^N)$ in $W^{N,1}(\mathbb{R}^N)$.
Corollary: Suppose that Ω is an open of class C^1 with bounded boundary $\partial\Omega$ or $\Omega = \mathbb{R}_+^N$. Let $1 \leq p \leq +\infty$; so

1. If $1 \leq p < N$ then

$$W^{1,p}(\Omega) \subset L^{P^*}(\Omega), \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$$

.

2. If $p = N$ then

$$W^{1,p}(\Omega) \subset L^q(\Omega), \quad \forall q \in [p, +\infty).$$

3. If $p > N$ then

$$W^{1,p}(\Omega) \subset L^\infty(\Omega).$$

Moreover, for $p > N$, we have for $u \in W^{1,p}(\Omega)$

$$|u(x) - u(y)| \leq C||u||_{W^{1,p}}|x - y|^\alpha, \text{ for almost } x, y \in \Omega$$

where

$$\alpha = 1 - \frac{N}{p} \text{ and } C = C(\Omega, p, N).$$

In particular

$$W^{1,p}(\Omega) \subset C(\bar{\Omega})$$

Proof. We extend u to $\mathbb{R}^N$ by the extension operator we then apply the above corollary to Pu.
Corollary: For $m \geq 2$ and $1 \leq p < \infty$ and Ω of class C^m we have the same embedding result for $W^{m,p}(\Omega)$ as in the case of $\Omega = \mathbb{R}^N$.
Theorem (Rellich Kondrachov): Suppose that Ω is bounded and of class C^1. So for

1. $p < N, \quad W^{1,p}(\Omega) \subset L^q(\Omega), \quad \forall q \in [1, p^*), \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$
2. $p = N, \quad W^{1,p}(\Omega) \subset L^q(\Omega), \quad \forall q \in [1, +\infty)$
3. $p > N, \quad W^{1,p}(\Omega) \subset C(\bar{\Omega})$

with compact embedding
Remarks:

1. If Ω is not bounded, the embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is not compact in general.

Example. on $[0, +\infty)$ let

$$f_n(x) = \left[\begin{array}{cc} x - (n-1), & n-1 < x \leq n \\ -x + (n+1), & n < x < n+1 \\ 0 & \text{otherwise} \end{array} \right.$$

$$\int_0^\infty |f_n| = 1 \text{ and } \int_0^\infty |f_n'(x)| = 2, \quad \forall n = 1, 2, \ldots$$

So

$$||f_n||_{W^{1,1}} = 3$$

$$f(x) = \lim_{n\to\infty} f_n(x) = 0, \quad \forall x \in [0, +\infty).$$

However, for any subsequence (f_{n_k})

$$\int_0^{+\infty} |f_{n_k} - f| = \int_0^{+\infty} |f_{n_k}| = 1,$$

which shows that no subsequence would converge in L^1. Thus the embedding is not compact.

2. The embedding of $W^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$ is never compact even if Ω is bounded and regular.

3. For the case $p = N$, the embedding of $W^{1,N}(\Omega)$ in $L^\infty(\Omega)$ is not always true even if Ω is bounded and of class C^1.

Example: Let

$$\Omega = \left\{(x, y) \in \mathbb{R}^2/x^2 + y^2 < \frac{1}{2}\right\}$$

and

$$u(x, y) = \left(L \log \frac{1}{x^2 + y^2}\right)^\alpha, \quad 0 < \alpha < \frac{1}{2}.$$

It is clear that $u \notin L^\infty(\Omega)$ because of the singularity at $(0, 0)$. However, $u \in W^{1,2}(\Omega)$ since

$$\begin{aligned}
\int_\Omega |u|^2 dx\, dy &= \int_0^{2\pi} \int_0^{\frac{1}{2}} \left(2\, L \log \frac{1}{r}\right)^{2\alpha} r\, dr\, d\theta \\
&= 2\pi \int_0^{\frac{1}{2}} 2^{2\alpha} \left(\log \frac{1}{r}\right)^{2\alpha} r\, dr \\
&= C\left[\int_0^{-e^{-1}} \left(L \log \frac{1}{r}\right)^{2\alpha} r\, dr + \int_{e^{-1}}^{\frac{1}{2}} \left(\log \frac{1}{r}\right)^{2\alpha} r\, dr\right]
\end{aligned}$$

The second integral is proper and has no problem. On $[0, e^{-1}]$, we have

$$\log \frac{1}{r} \geq 1 \Rightarrow \left(\log \frac{1}{r}\right)^{2\alpha} \leq \log \frac{1}{r} \text{ since } 2\alpha < 1.$$

Thus

$$\begin{aligned}\int_0^{e^{-1}} \left(\log \frac{1}{r}\right)^{2\alpha} dr &\leq -\int_0^{e^{-1}} (+\log r) r\, dr \\ &\leq -\left[\frac{r^2}{2} \log r|_0^{e^{-1}} - \int_0^{e^{-1}} \frac{r^2}{2}\frac{1}{r} dr\right] \\ &\leq \frac{e^{-2}}{2} + \frac{e^{-2}}{4} = \frac{3}{4}e^{-2} < \infty\end{aligned}$$

Consequently

$$\int_\Omega |u|^2 dx\, dy < \infty.$$

It is easy to see that

$$u_x = -\alpha \frac{2x}{x^2+y^2}\left(-\log(x^2+y^2)\right)^{\alpha-1}$$

Therefore

$$\int_\Omega |u_x|^2 = 2^{2x}\alpha^2 \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^{\frac{1}{2}} \frac{(-\log r)^{2\alpha-2}}{r}\, dr$$

we make the change of variable $t = \frac{1}{r}$ to get

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{(-\log r)^{2\alpha-2}}{r} dr &= \int_2^\infty \frac{(\log\ t)^{2\alpha-2}}{t}\, dt \\ &= \left.\frac{(\log t)^{2\alpha-1}}{2\alpha-1}\right|_{t=0}^{t=\infty} = \frac{(\log\ 2)^{2\alpha-1}}{1-2\alpha}.\end{aligned}$$

since $2\alpha - 1 < 0$. Thus $\int_\Omega |u_x|^2 < \infty$ and $\int_\Omega |u_y|^2 < \infty$ by similar computations.