

ON THE CONTROL OF SOLUTIONS OF A VISCOELASTIC EQUATION

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ABSTRACT

In this paper we consider the viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau = 0,$$

in a bounded domain. We show that the damping caused by the integral term is strong enough to control and stabilise the solution. Precisely, we establish an exponential decay result.

1. INTRODUCTION

In [1], Cavalcanti *et al.* studied

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau \\ + a(x)u_t + |u|^\gamma u = 0, \text{ in } \Omega \times (0, \infty) \\ u(x, t) = 0, x \in \partial\Omega, t \geq 0 \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \end{cases} \quad (1)$$

where Ω is a bounded domain of \mathbb{R}^n ($n \geq 1$) with a smooth boundary $\partial\Omega$, $\gamma > 0$, g is a positive function, and $a : \Omega \rightarrow \mathbb{R}^+$ is a function, which may be null on a part of Ω . Under the condition that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), t \geq 0,$$

such that $\|g\|_{L^1((0, \infty))}$ is small enough, the authors obtained an exponential rate of decay. This work extended the result of Zuazua [2], in which he considered (1) with $g = 0$ and the linear damping is localized. Berrimi and Messaoudi [3] improved Cavalcanti's result by introducing a different functional, which allows them to weaken the conditions on both a and g . In particular, the function a can vanish on the whole domain Ω and consequently the geometry condition has disappeared. In [4], Cavalcanti *et al* considered

$$\begin{cases} u_{tt} - k_0 \Delta u + \int_0^t \text{div}[a(x)g(t - \tau) \nabla u(\tau)] d\tau \\ + b(x)h(u_t) + f(u) = 0, \text{ in } \Omega \times (0, \infty), \end{cases}$$

under similar conditions on the relaxation function g and $a(x) + b(x) \geq \delta > 0$, for all $x \in \Omega$. They improved the result in [1] by establishing exponential stability for g decaying exponentially and h linear and polynomial stability for g decaying polynomially and h nonlinear. Their proof, based on the use of piecewise multipliers, is similar to the one in [1]. Though both results in [3] and [4] improve the earlier one in [1], the approaches are different Another

problem, where the damping induced by the viscosity is acting on the domain and a part of the boundary, was also discussed by Cavalcanti *et al* [5] and existence and uniform decay rate results were established. In the same direction, Cavalcanti *et al* [6] have also studied, in a bounded domain, the following equation

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t - \tau) \Delta u(\tau) d\tau \\ - \gamma \Delta u_t = 0, \quad x \in \Omega, \quad t > 0, \end{cases}$$

$\rho > 0$, and proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. This last result has been extended to a situation, where a source term is competing with the strong mechanism damping and the one induced by the viscosity, by Messaoudi and Tatar [7]. In their work, Messaoudi and Tatar combined the well depth method with the perturbation techniques to show that solution with positive, but small, energy exist globally and decay to the rest exponentially. Also, Messaoudi [8] considered

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau \\ + a u_t |u_t|^m = b |u|^\gamma u, \text{ in } \Omega \times (0, \infty) \end{cases}$$

and showed, under suitable conditions on g , that solutions with negative energy blow up in finite time if $\gamma > m$ and continue to exist if $m \geq \gamma$. We also should mention the work of Kavashima and Shibata [9], in which a global existence and exponential stability of small solutions to a nonlinear viscoelastic problem has been established.

In the present work, we are concerned with

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau = 0, \\ u(x, t) = 0, x \in \partial\Omega \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \end{cases} \quad (2)$$

$(x, t) \in \Omega \times (0, \infty)$. We will show that the damping caused by the integral term is enough to obtain an exponential decay result under weaker conditions on the relaxation function g . Our choice of the 'Lyapounov' functional made our proof easier than the one in [1] and [4].

2. PRELIMINARIES

In this section, we present the conditions on the control function g and state, without proof, a global existence result, which may be proved by repeating the argument of [1]. So we assume

$g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded C^1 function such that

$$\begin{aligned} g(0) &> 0, \quad 1 - \int_0^\infty g(s)ds = l > 0, \\ g'(t) &\leq -\xi g(t), \quad t \geq 0, \end{aligned} \quad (3)$$

for a positive constant ξ .

Proposition 2.1 Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that g satisfies $1 - \int_0^\infty g(s)ds = l > 0$. Then problem (2) has a unique global solution

$$\begin{aligned} u &\in C([0, \infty); H_0^1(\Omega)) \\ u_t &\in C([0, \infty); L^2(\Omega)). \end{aligned} \quad (4)$$

Next, we introduce

$$\begin{aligned} \mathcal{E}(t) &:= \frac{1}{2} \left(1 - \int_0^t g(s)ds \right) \|\nabla u(t)\|_2^2 \\ &\quad + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t), \end{aligned} \quad (5)$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau) \|v(t) - v(\tau)\|_2^2 d\tau. \quad (6)$$

3. EXPONENTIAL DECAY

In this section, we state and prove our main result. We start with the following

Lemma 3.1 The "modified" energy satisfies

$$\begin{aligned} \mathcal{E}'(t) &= \frac{1}{2} (g' \circ \nabla u)(t) \\ -\frac{1}{2} g(t) \|\nabla u(t)\|_2^2 &\leq \frac{1}{2} (g' \circ \nabla u)(t) \leq 0. \end{aligned} \quad (7)$$

For the proof of this lemma, see [1] or [8] for instance.

Theorem 3.2 Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that g satisfies (3). Then for any $t_0 > 0$ there exist positive constants k and K such that the solution given by (4) satisfies

$$\mathcal{E}(t) \leq K e^{-kt}, \quad \forall t \geq t_0 > 0. \quad (8)$$

Proof We define

$$F(t) := \mathcal{E}(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t) \quad (9)$$

where ε_1 and ε_2 are positive constants to be specified later and

$$\begin{aligned} \Psi(t) &:= \int_\Omega uu_t dx, \\ \chi(t) &:= - \int_\Omega u_t \int_0^t g(t-\tau) (u(t) - u(\tau)) d\tau dx. \end{aligned}$$

It is straightforward to see that for ε_1 and ε_2 so small, we have

$$\alpha_1 F(t) \leq \mathcal{E}(t) \leq \alpha_2 F(t), \quad (10)$$

holds for two positive constants α_1 and α_2 .

By using equation (2), we easily see that

$$\begin{aligned} \Psi'(t) &= \int_\Omega (uu_{tt} + u_t^2) dx = \int_\Omega u_t^2 dx - \int_\Omega |\nabla u|^2 dx \\ &\quad + \int_\Omega \nabla u(t) \cdot \int_0^t g(t-\tau) \nabla u(\tau) d\tau dx \end{aligned} \quad (11)$$

We now estimate the third term in the right-hand side of (11) as follow

$$\begin{aligned} &\int_\Omega \nabla u(t) \cdot \int_0^t g(t-\tau) \nabla u(\tau) d\tau dx \\ &\leq \frac{1}{2} \int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2} \int_\Omega \left[\int_0^t g(t-\tau) \times \right. \\ &\quad \left. (|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau \right]^2 dx \end{aligned} \quad (12)$$

We then use Cauchy-Schwarz inequality, Young's inequality, and the fact that

$$\int_0^t g(\tau) d\tau \leq \int_0^\infty g(\tau) d\tau = 1 - l,$$

to obtain, for any $\eta > 0$,

$$\begin{aligned} &\int_\Omega \left[\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau \right]^2 dx \\ &\leq \int_\Omega \left[\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) d\tau \right]^2 dx \\ &\quad + \int_\Omega \left[\int_0^t g(t-\tau) |\nabla u(t)| d\tau \right]^2 dx \\ &\quad + 2 \int_\Omega \left[\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) d\tau \right] \times \\ &\quad \left[\int_0^t g(t-\tau) |\nabla u(t)| d\tau \right] dx \\ &\leq (1 + \eta) \int_\Omega \left[\int_0^t g(t-\tau) |\nabla u(t)| d\tau \right]^2 dx + \\ &\quad (1 + \frac{1}{\eta}) \int_\Omega \left[\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) d\tau \right]^2 dx \\ &\leq (1 + \eta) (1 - l)^2 \int_\Omega |\nabla u(t)|^2 dx \\ &\quad + (1 + \frac{1}{\eta}) (1 - l) (g \circ \nabla u)(t) \end{aligned} \quad (13)$$

By combining (11) - (13), we have

$$\begin{aligned} \Psi'(t) &\leq \int_\Omega u_t^2 dx + \frac{1}{2} (1 + \frac{1}{\eta}) (1 - l) \times \\ &\quad (g \circ \nabla u)(t) - \frac{1}{2} \{ 1 - (1 + \eta) \} \\ &\quad \times (1 - l)^2 \int_\Omega |\nabla u(t)|^2 dx \end{aligned} \quad (14)$$

By choosing $\eta = l/(1 - l)$, (14) becomes

$$\begin{aligned} \Psi'(t) &\leq \int_\Omega u_t^2 dx - \frac{l}{2} \int_\Omega |\nabla u|^2 dx \\ &\quad + \frac{(1 - l)}{2l} (g \circ \nabla u)(t) \end{aligned} \quad (15)$$

Next we estimate

$$\begin{aligned}
 \chi'(t) &= - \int_{\Omega} u_{tt} \int_0^t g(t-\tau)(u(t) - u(\tau))d\tau dx \\
 &- \int_{\Omega} u_t \int_0^t g'(t-\tau)(u(t) - u(\tau))d\tau dx \\
 &- \left(\int_0^t g(s)ds \right) \int_{\Omega} u_t^2 dx = \int_{\Omega} \nabla u(t) \times \\
 &\left(\int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))d\tau \right) dx \\
 &- \int_{\Omega} \left(\int_0^t g(t-\tau)\nabla u(\tau)d\tau \right) \times \\
 &\left(\int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))d\tau \right) dx \\
 &- \int_{\Omega} u_t \int_0^t g'(t-\tau)(u(t) - u(\tau))d\tau dx \\
 &- \left(\int_0^t g(s)ds \right) \int_{\Omega} u_t^2 dx
 \end{aligned} \tag{16}$$

In a similar way, we estimate the right-hand side terms of (16). So for $\delta > 0$, we have :

The first term

$$\begin{aligned}
 &\int_{\Omega} \nabla u(t) \cdot \left(\int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))d\tau \right) dx \\
 &\leq \delta \int_{\Omega} |\nabla u|^2 dx + \frac{(1-l)}{4\delta} (g \circ \nabla u)(t).
 \end{aligned} \tag{17}$$

The second term

$$\begin{aligned}
 &- \int_{\Omega} \left(\int_0^t g(t-s)\nabla u(s)ds \right) \times \\
 &\left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds \right) dx \\
 &\leq \delta \int_{\Omega} \left| \int_0^t g(t-s)\nabla u(s)ds \right|^2 dx \\
 &+ \frac{1}{4\delta} \int_{\Omega} \left| \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds \right|^2 dx \\
 &\leq \delta \int_{\Omega} \left[\int_0^t g(t-s)(|\nabla u(t) - \nabla u(s)| \right. \\
 &+ |\nabla u(t)|)ds \left. \right]^2 dx + \frac{(1-l)}{4\delta} (g \circ \nabla u)(t) \\
 &\leq 2\delta \int_{\Omega} \left[\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|ds \right]^2 dx \\
 &+ 2\delta \int_{\Omega} \left[\int_0^t g(t-s)\nabla u(t)ds \right]^2 dx \\
 &+ \frac{(1-l)}{4\delta} (g \circ \nabla u)(t) \leq +2\delta(1-l)^2 \int_{\Omega} |\nabla u|^2 dx \\
 &(2\delta + \frac{1}{4\delta})(1-l)(g \circ \nabla u)(t)
 \end{aligned} \tag{18}$$

The third term

$$\begin{aligned}
 &- \int_{\Omega} u_t \int_0^t g'(t-\tau)(u(t) - u(\tau))d\tau dx \\
 &\leq \delta \int_{\Omega} |u_t|^2 dx + \frac{1}{4\delta} \left(\int_0^t -g'(t-s)ds \right) \times \\
 &\int_{\Omega} \int_0^t -g'(t-s)|u(t) - u(s)|^2 ds dx \\
 &\leq \delta \int_{\Omega} |u_t|^2 dx + \frac{g(0)}{4\delta} C_p \int_{\Omega} \int_0^t -g'(t-s) \times \\
 &|\nabla u(t) - \nabla u(s)|^2 ds dx,
 \end{aligned} \tag{19}$$

where C_p is the embedding constant. A combination of (16) - (19) then yields

$$\begin{aligned}
 \chi'(t) &\leq \delta \{1 + 2(1-l)^2\} \|\nabla u\|_2^2 \\
 &+ \{2\delta + \frac{1}{2\delta}\} (1-l)(g \circ \nabla u)(t) \\
 &+ \frac{g(0)}{4\delta} C_p (-g' \circ \nabla u)(t) \\
 &+ \left\{ \delta - \int_0^t g(s)ds \right\} \int_{\Omega} u_t^2 dx,
 \end{aligned} \tag{20}$$

Since $g(0) > 0$ then there exists $t_0 > 0$ such that

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0, \tag{21}$$

$\forall t \geq t_0$. By using (7), (9), (3.9), (20) and (21) we obtain

$$\begin{aligned}
 F'(t) &\leq -[\varepsilon_2 \{g_0 - \delta\} - \varepsilon_1] \int_{\Omega} u_t^2 dx \\
 &- \left[\frac{\varepsilon_1 l}{2} - \varepsilon_2 \delta \{1 + 2(1-l)^2\} \right] \|\nabla u\|_2^2 \\
 &+ \left\{ \frac{1}{2} - \frac{\varepsilon_1(1-l)}{2\varepsilon l} - \varepsilon_2 \left\{ \frac{g(0)}{4\delta} C_p + \right. \right. \\
 &\left. \left. + \frac{(1-l)}{\varepsilon} \left(2\delta + \frac{1}{2\delta} \right) \right\} \right\} (g' \circ \nabla u)(t).
 \end{aligned} \tag{22}$$

At this point we choose δ so small that

$$g_0 - \delta > \frac{1}{2}g_0, \quad \frac{1}{l}\delta \{1 + 2(1-l)^2\} < \frac{1}{8}g_0.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{1}{4}g_0\varepsilon_2 < \varepsilon_1 < \frac{1}{2}g_0\varepsilon_2 \tag{23}$$

will make

$$\begin{aligned}
 k_1 &= \varepsilon_2 \{g_0 - \delta\} - \varepsilon_1 > 0, \\
 k_2 &= \frac{\varepsilon_1 l}{2} - \varepsilon_2 \delta \{1 + 2(1-l)^2\} > 0.
 \end{aligned}$$

We then pick ε_1 and ε_2 so small that (10) and (23) remain valid and

$$\frac{1}{2} - \frac{\varepsilon_1(1-l)}{2\varepsilon l} - \varepsilon_2 \left\{ \frac{g(0)}{4\delta} C_p + \frac{(1-l)}{\varepsilon} \left(2\delta + \frac{1}{2\delta} \right) \right\} > 0 \tag{24}$$

Therefore (24) becomes

$$F'(t) \leq -\beta \mathcal{E}(t) \leq -\beta \alpha_1 F(t), \quad \forall t \geq t_0. \tag{25}$$

by virtue of (10). A simple integration of (25) leads to

$$F(t) \leq F(t_0) e^{\beta \alpha_1 t_0} e^{-\beta \alpha_1 t}, \quad \forall t \geq t_0. \tag{26}$$

Again by(10), estimate (26) yields

$$\mathcal{E}(t) \leq \alpha_2 F(t_0) e^{\beta \alpha_1 t_0} e^{-\beta \alpha_1 t}, \quad \forall t \geq t_0. \tag{27}$$

This completes the proof.

4. ACKNOWLEDGMENT

This work has been funded by King Fahd University of Petroleum and Minerals under Project Number MS/VISCO ELASTIC/270. The author would like to express his sincere thanks to KFUPM for its support.

5. REFERENCES

- [1] Cavalcanti M. M., V. N. Domingos Cavalcanti and J. A. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, *Elect. J. Diff. Eqs.* Vol. **2002** (2002) 44, 1 - 14.
- [2] Zuazua E., Exponential decay for the semilinear wave equation with locally distributed damping, *Comm. PDE.* **15** (1990), 205 - 235.
- [3] Berrimi S. and Messaoudi S. A., Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping, *Elect J. Diff. Eqns.* vol **2004 # 88** (2004), 1- 10.
- [4] Cavalcanti M. M. and Oquendo H. P., Frictional versus viscoelastic damping in a semilinear wave equation, *SIAM J. Control Optim.* vol **42 # 4** (2003), 1310 - 1324.
- [5] Cavalcanti M. M., V. N. Domingos Cavalcanti, J. S. Prates Filho and J. A. Soriano, Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping, *Diff. Integ. Eqs.* **14** (2001) 1, 85 - 116.
- [6] Cavalcanti M. M., V. N. Domingos Cavalcanti and J. Ferreira, Existence and uniform decay for nonlinear viscoelastic equation with strong damping, *Math. Meth. Appl. Sci.* **24** (2001), 1043 - 1053.
- [7] Messaoudi S. A. and Tatar N-E., Global existence asymptotic behavior for a Nonlinear Viscoelastic Problem, *Math. Meth. Sci. Research J.* Vol. **7 # 4** (2003), 136-149.
- [8] Messaoudi S. A., Blow up and global existence in a nonlinear viscoelastic wave equation, *Mathematische Nachrichten* vol. **260** (2003), 58 - 66.
- [9] Kawashima S. and Shibata Y., Global Existence and exponential stability of small solutions to nonlinear viscoelasticity, *Comm. Math. Physics* Vol. **148 # 1**(1992), 189 - 208.