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ENERGY DECAY OF SOLUTIONS OF A SEMILINEAR WAVE EQUATION

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Abstract: We consider the multi-dimensional semilinear wave equation

$$u_{tt} - \Delta u = -\alpha u_t + \nabla\Phi(x).\nabla u - \beta|u|^{p-2}u,$$

$\alpha, \beta > 0$, associated with initial-boundary conditions. We first prove a local existence theorem for arbitrary initial data. We then show that this solution is global with an energy that decays exponentially to zero.

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1. Introduction

In [15], Pucci and Serrin studied the following problem

$$\begin{aligned} u_{tt} - \Delta u + Q(x,t,u,u_t) + f(x,u) &= 0, \quad x \in \Omega, \quad t > 0, \\ u(x,t) &= 0, \quad x \in \partial\Omega, \ t \geq 0, \\ u(x,0) = \phi(x), \quad u_t(x,0) &= \varphi(x), \ x \in \Omega, \end{aligned} \tag{1.1}$$

and proved that the energy of the solution is a Lyaponov function. Although, they did not discuss the issue of the decay rate, they did show that in general this energy goes to zero as t approaches infinity. They also considered an important special case which occurs when $Q(x,t,u,u_t) = a(t)t^\alpha u_t$ and $f(x,u) = V(x)u$ and showed that the behavior of the solutions depends crusially on the parameter α. If $|\alpha| \leq 1$, then the rest field is asymptotically stable.

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On the other hand, when $\alpha < -1$ or $\alpha > 1$, there are solutions that do not approach zero or they approach nonzero functions $\phi(x)$ as $t \to \infty$.

In [14], Nakao studied (1.1) in an abstract setting and established a theorem concerning the decay of the solution energy. His result shows that the energy decays exponentially for the linear damping case ($Q(x,t,u,u_t) = au_t$) and it decays in the rate of $t^{-2/m-2}$ when $Q(x,t,u,u_t) = a|u_t|^{m-2}u_t$, $m > 2$.

In [7] and also in [16], the linear wave equation associated with a nonlinear feedback at the boundary has been considered. Precisely, the authors looked into the following problem

$$\begin{aligned}
&u_{tt} - \Delta u = 0, && x \in \Omega, \quad t > 0,\\
&\frac{\partial u}{\partial \nu}(x,t) = -m(x).\nu(x)g(u_t), && x \in \Gamma_0, \quad t > 0,\\
&u(x,t) = 0, && x \in \Gamma_1, \quad t > 0,\\
&u(x,0) = \phi(x), \quad u_t(x,0) = \varphi(x), && x \in \Omega,
\end{aligned}$$

where $m(x) = x - x_0$, $x_0 \in \mathbb{R}^n, \Gamma_0 = \{x \in \partial\Omega \ : \ m(x).\nu(x) > 0\}$, and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, with $\Gamma_1 \neq \phi$. They discussed the rate of decay of the energy of the solution and established, under certain growth conditions on g, a similar result to [14].

In this article, we deal with the energy decay of the solution for the initial boundary value problem

$$\begin{aligned}
&u_{tt} - \Delta u = -\alpha u_t + \nabla\Phi(x).\nabla u - \beta|u|^{p-2}u, && x \in \ \Omega, \quad t > 0,\\
&u(x,t) = 0, && x \in \partial\Omega, \quad t > 0,\\
&u(x,0) = \phi(x), \quad u_t(x,0) = \varphi(x), && x \in \ \Omega,
\end{aligned} \tag{1.2}$$

where α and β are strictly positive constants, $p > 2$, $\Phi \in W^{1,\infty}(\Omega)$, and Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$.

For $\Phi = 0$ and $\beta = 0$, it is well known that the damping term αu_t assures global existence for arbitrary initial data (see [4], [8]). If $\Phi = 0$, $\alpha = 0$ and $\beta < 0$ then the source term $-\beta u|u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [2], [5], [9], [10]). The interaction between the damping term and the source term has been first considered by Levine [9], [10]. For $\Phi = 0$, $\alpha > 0$, and $\beta < 0$, the author showed that solutions with negative initial energy blow up in finite time. This result has been extended to the situation where $\Phi \neq 0$ by Messaoudi [13].

For Φ small in L^∞ norm, (1.2) is a special case of (1.1). However, to my knowledge, no result concerning the energy decay of this problem has been discussed for arbitrary Φ in $W^{1,\infty}(\Omega)$. To accomplish this goal, we use an argument close to the one presented by Aassila and Guesmia [1]. This argument is based on a theorem by Komornik [6], which we state as a lemma without proof. This work is divided into two parts. In part one, we establish a local existence theorem. In part two, we show that this local solution is, in fact, global with energy that decays exponentially to zero.

2. Local Existence

First let us consider, for v given, the linear problem

$$\begin{aligned} u_{tt} - \Delta u &= -\alpha u_t + \nabla\Phi(x).\nabla u - \beta |v|^{p-2} v, && x \in \Omega, \quad t > 0, \\ u(x,t) &= 0, && x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \phi(x), &\quad u_t(x,0) = \varphi(x), && x \in \Omega, \end{aligned} \tag{2.1}$$

where u is the sought solution.

Lemma 2.1. *Assume that $\Phi \in W^{1,\infty}(\Omega)$ and $p > 2$ with*

$$p \leq 2\frac{n-1}{n-2}, \tag{2.2}$$

if $n \geq 3$. Then given any v in $C([0,\ T];\ C_0^\infty(\Omega))$ and ϕ, φ in $C_0^\infty(\Omega)$, the problem (2.1) has a unique solution

$$u \in W^{j,\infty}\left((0,\ T);\ H^j(\Omega) \cap H_0^1(\Omega)\right), \quad j = 0, 1, 2.. \tag{2.3}$$

This lemma is a direct result of Theorem 3.1, Chapter 1, [12]. It can also be established by using a classical energy argument (see [4] for instance).

Lemma 2.2. *Let the assumptions of Lemma 2.1 hold. Then given any ϕ in $H_0^1(\Omega)$, φ in $L^2(\Omega)$, and v in $C([0,\ T];\ H_0^1(\Omega))$, the problem (2.1) has a unique solution*

$$u \in C\left([0,\ T];\ H_0^1(\Omega)\right) \cap C^1\left([0,\ T];\ L^2(\Omega)\right) \cap C^2\left([0,\ T];\ H^{-1}(\Omega)\right). \tag{2.4}$$

Moreover, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega e^{\Phi(x)}[u_t^2 + |\nabla u|^2](x,t)dx + \int_0^t \int_\Omega e^{\Phi(x)} u_t^2(x,s)dxds \\ &= \frac{1}{2}\int_\Omega e^{\Phi(x)}[u_1^2 + |\nabla u_0|^2](x)dx + \beta \int_0^t \int_\Omega e^{\Phi(x)} |v|^{p-2} v u_t(x,s)dxds, \end{aligned} \tag{2.5}$$

$\forall t \in [0,\ T]$.

Proof. We approximate ϕ, φ by sequences (ϕ^μ), $(\varphi^\mu) \subset C_0^\infty(\Omega)$, and v by a sequence $(v^\mu) \subset C\left([0,\ T];\ C_0^\infty(\Omega)\right)$. We then consider the linear problem

$$\begin{aligned} u_{tt}^\mu - \Delta u^\mu &= -\alpha u_t^\mu + \nabla\Phi(x).\nabla u^\mu - \beta|v^\mu|^{p-2}v^\mu, \quad x \in \Omega, \quad t > 0, \\ u^\mu(x,t) &= 0, \qquad\qquad x \in \partial\Omega, \quad t > 0, \qquad (2.6) \\ u^\mu(x,0) = \phi^\mu(x), &\quad u_t^\mu(x,0) = \varphi^\mu(x), \qquad x \in \Omega. \end{aligned}$$

Lemma 2.1 guarantees the existence of a unique solution u^μ satisfying (2.3). Now we procced to show that the sequence of solutions (u^μ) is Cauchy in

$$\mathbf{W} := C\left([0,\ T];\ H_0^1(\Omega)\right) \cap C^1\left([0,\ T];\ L^2(\Omega)\right), \qquad (2.7)$$

equipped with the norm

$$||w||_{\mathbf{W}}^2 := \max\left\{\int_\Omega [w_t^2 + |\nabla w|^2](x,t)dx,\ 0 \le t \le T\right\}.$$

For this aim, we set

$$U := u^\mu - u^\nu, \qquad V := v^\mu - v^\nu.$$

It is straightforward to see that U satisfies

$$\begin{aligned} U_{tt} - \Delta U &= -\alpha U_t + \nabla\Phi(x).\nabla U - \beta|v^\mu|^{p-2}v^\mu + \beta|v^\nu|^{p-2}v^\nu, \\ U(x,t) &= 0, \quad x \in \partial\Omega, \quad t > 0, \qquad (2.8) \\ U(x,0) = U_0(x) &= \phi^\mu(x) - \phi^\nu(x),\ U_t(x,0) = U_1(x) = \varphi^\mu(x) - \varphi^\nu(x). \end{aligned}$$

We multiply equation (2.8) by $e^{\Phi(x)}U_t$ and integrate over Ω x $(0,\ t)$ to get

$$\begin{aligned} \frac{1}{2}\int_\Omega e^{\Phi(x)}[U_t^2 + |\nabla U|^2](x,t)dx + \alpha\int_0^t\int_\Omega e^{\Phi(x)}U_t^2(x,s)dxds \\ = \frac{1}{2}\int_\Omega e^{\Phi(x)}[U_1^2 + |\nabla U_0|^2](x)dx \\ - \beta\int_0^t\int_\Omega .e^{\Phi(x)}[|v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu]U_t(x,s)dxds. \quad (2.9) \end{aligned}$$

This, in turns, yields

$$\begin{aligned} \frac{1}{2}\int_\Omega [U_t^2 + |\nabla U|^2](x,t)dx + \alpha\int_0^t\int_\Omega U_t^2(x,s)dxds \le \frac{1}{2}\frac{A}{a}\int_\Omega [U_1^2 + |\nabla U_0|^2](x)dx \\ + \beta\frac{A}{a}\int_0^t\int_\Omega \left|[|v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu]U_t(x,s)\right| dxds, \quad (2.10) \end{aligned}$$

where a and A satisfy

$$a \leq e^{\Phi(x)} \leq A, \qquad \forall\, x \in \Omega.$$

We then estimate the last term in (2.10) as follows

$$\int_\Omega \left| [|v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu] U_t(x,s) \right| dx$$
$$\leq C\|U_t\|_2 \|V\|_{2n/(n-2)} \left[\|v^\mu\|_{n(p-2)}^{p-2} + \|v^\nu\|_{n(p-2)}^{p-2} \right]. \quad (2.11)$$

The Sobolev embedding and condition (2.2) give

$$\|V\|_{2n/(n-2)} \leq C\|\nabla V\|_2, \quad \|v^\mu\|_{n(p-2)}^{p-2} + \|v^\nu\|_{n(p-2)}^{p-2}$$
$$\leq C\left[\|\nabla v^\mu\|_2^{p-2} + \|\nabla v^\nu\|_2^{p-2} \right],$$

where C is a constant depending on Ω only. Therefore (2.11) takes the form

$$\int_\Omega \left| [|v^\mu|^{p-2}v^\mu - |v^\nu|^{p-2}v^\nu] U_t(x,s) \right| dx$$
$$\leq C\|U_t\|_2 \|\nabla V\|_2 \left[\|\nabla v^\mu\|_2^{p-2} + \|\nabla v^\nu\|_2^{p-2} \right];$$

hence (2.10) becomes

$$\frac{1}{2}\int_\Omega [U_t^2 + |\nabla U|^2](x,t)dx \leq \frac{A}{2a}\int_\Omega [U_1^2 + |\nabla U_0|^2](x)dx$$
$$+ \Gamma \int_0^t \|U_t(..,s)\|_2 \|\nabla V(..,s)\|_2 ds,$$

where Γ is a generic positive constant depending on a, A, Ω, and the radius of the ball in $C([0,\ T];\ H_0^1(\Omega))$ containing v^μ and v^ν. Young's inequality then guarantees

$$\|U\|_{\mathbf{W}}^2 \leq \Gamma \int_\Omega [U_1^2 + |\nabla U_0|^2](x)dx + \Gamma T\ \|V\|_{\mathbf{W}}^2.$$

Since (ϕ^μ) is Cauchy in $H_0^1(\Omega)$, (φ^μ) is Cauchy in $L^2(\Omega)$, and (v^μ) is Cauchy in $C([0,\ T];\ H_0^1(\Omega))$, we conclude that that (u^μ) is Cauchy in $\mathbf{W}$; hence (u_t^μ) is Cauchy in $L^2((\Omega) \times (0,\ t))$. Therefore (u^μ) converges to a limit u in $\mathbf{W}$. We now show that this limit u is a weak solution of (2.1) in the sense of [11]. That is for each θ in $H_0^1(\Omega)$, we must show that

$$\frac{d}{dt}\int_\Omega u_t(x,t)\theta dx + \int_\Omega \nabla u(x,t).\nabla\theta(x)dx = -\alpha\int_\Omega u_t(x,t)\theta(x)dx$$
$$+ \int_\Omega \nabla\Phi(x).\nabla u(x,t)\theta(x)dx - \beta\int_\Omega |v|^{p-2}v(x,t)\theta(x)dx, \quad (2.12)$$

for each t in $[0, T]$. To establish this, we multiply equation (2.6) by θ and integrate over Ω, so we obtain

$$\frac{d}{dt}\int_\Omega u_t^\mu(x,t)\theta dx = -\int_\Omega \nabla u^\mu(x,t).\nabla\theta dx - \alpha\int_\Omega u_t^\mu(x,t)\theta(x)dx + \int_\Omega \nabla\Phi(x).\nabla u^\mu(x,t)\theta(x)dx - \beta\int_\Omega |v^\mu|^{p-2}v^\mu(x,t)\theta(x)dx. \quad (2.13)$$

As $\mu\to\infty$, we see that each term in the righthand side of (2.13) is in $C([0, T])$. We thus have $\int_\Omega u_t(x,t)\theta dx$ $\{= lim \int_\Omega u_t^\mu(x,t)\theta dx\}$ is a C^1 function on $[0, T]$, so (2.12) holds for each t in $[0, T]$. For the energy equality (2.5), we start from the energy equality for u^μ and proceed in the same way to establish it for u. To prove uniqueness, we take v_1 and v_2 and let u_1 and u_2 be the corresponding solutions of (2.1). It is clear that $U = u_1 - u_2$ satisfies

$$\frac{1}{2}\int_\Omega e^{\Phi(x)}[U_t^2 + |\nabla U|^2](x,t)dx + \alpha\int_0^t\int_\Omega e^{\Phi(x)}U_t^2(x,s)dxds = -\beta\int_0^t\int_\Omega e^{\Phi(x)}[|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2]U_t(x,s)dxds\,. \quad (2.14)$$

If $v_1 = v_2$, then (2.14) shows that $U = 0$, which implies uniqueness. This completes the proof.

Remark 2.1. This result, as well as the results below, hold if $\Phi\in L^\infty(\Omega)$ with $\nabla\Phi$ defined almost everywhere. In this case the limit u is a weak solution of (2.6) in the following sense

$$\frac{d}{dt}\int_\Omega e^{\Phi(x)}u_t(x,t)\theta dx + \int_\Omega e^{\Phi(x)}\nabla u(x,t).\nabla\theta dx = -\alpha\int_\Omega e^{\Phi(x)}u_t(x,t)\theta(x)dx - \beta\int_\Omega e^{\Phi(x)}|v|^{p-2}v(x,t)\theta(x)dx,$$

for each θ in $H_0^1(\Omega)$.

Theorem 2.3. *Assume that $\Phi\in W^{1,\infty}(\Omega)$ and $p>2$, satisfying (2.2) if $n\geq 3$. Assume further that*

$$\phi\in H_0^1(\Omega),\qquad \varphi\in L^2(\Omega).$$

Then (1.2) has a unique solution

$$u\in C\left([0,\ T];\ H_0^1(\Omega)\right)\cap C^1\left([0,\ T];\ L^2(\Omega)\right), \quad (2.15)$$

T is small enough.

Proof. For $M > 0$ large and $T > 0$, we define a class of functions $Z(M,T)$ which consists of all functions w in $\mathbf{W}$ satisfying the initial conditions of (1.2) and

$$||w||^2_{\mathbf{W}} \le M^2. \tag{2.16}$$

$Z(M,T)$ is nonempty if M is large enough. This follows from the trace theorem (see [11]). We also define the map f from $Z(M,T)$ into $\mathbf{W}$ by $u := f(v)$, where u is the unique solution of the linear problem (2.1). We would like to show, for M sufficiently large and T sufficiently small, that f is a contraction from $Z(M,T)$ into itself. For this purpose, we use the energy equality (2.5), which yields

$$\begin{aligned}&\int_\Omega [u_t^2 + |\nabla u|^2](x,t)dx + 2\alpha \int_0^t \int_\Omega u_t^2(x,s)dxds \\ &\le \frac{A}{a}\int_\Omega [u_1^2 + |\nabla u_0|^2](x)dx + \beta \frac{A}{a}\int_0^t \int_\Omega |v|^{p-1}|u_t|(x,s)dxds, \quad \forall\, t \in [0,\ T] \\ &\le \frac{A}{a}\int_\Omega [u_1^2 + |\nabla u_0|^2](x)dx + \beta C\frac{A}{a}\int_0^t ||u_t||_2 ||\nabla v||_2^{p-1}, \quad \forall\, t \in [0,\ T].\end{aligned}$$

This leads to

$$||u||^2_{\mathbf{W}} \le C\int_\Omega [u_1^2 + |\nabla u_0|^2](x)dx + CM^{p-1}T||u||_{\mathbf{W}},$$

where C is independant of M. By choosing M large enough and T sufficiently small, (2.16) is satisfied; hence $u \in Z(M,T)$.

Next we verify that f is a contraction. To this end we set $U = u_1 - u_2$ and $V = v_1 - v_2$, where $u_1 = f(v_1)$ and $u_2 = f(v_2)$. It is straightforward to verify that U satisfies

$$\begin{aligned}U_{tt} - \Delta U &= -\alpha U_t + \nabla\Phi(x).\nabla U - \beta|v_1|^{p-2}v_1 + \beta|v_2|^{p-2}v_2\,, \\ U(x,t) &= 0, \quad x \in \partial\Omega, \quad t > 0\,, \\ U(x,0) &= U_t(x,0) = 0, \quad x \in \Omega\,.\end{aligned} \tag{2.17}$$

By multiplying equation (2.17) by $e^{\Phi(x)}U_t$ and integrating over Ω x $(0,\, t)$, we arrive at

$$\begin{aligned}\int_\Omega [U_t^2 + |\nabla U|^2](x,t)dx &\le \frac{A}{a}\int_0^t\int_\Omega \left||v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\right| |U_t|\,(x,s)dxds \\ &\le C\frac{A}{a}\int_0^t ||U_t||_2||\nabla V||_2(||\nabla v_1||_2^{p-2} + ||\nabla v_2||_2^{p-2})(.,s)ds\,.\end{aligned}$$

Thus we have

$$||U||^2_{\mathbf{W}} \le \frac{A}{a}CTM^{p-2}||V||^2_{\mathbf{W}}. \tag{2.18}$$

By choosing T so small that $CTM^{p-2}A/a < 1$, (2.18) shows that f is a contraction. The contraction mapping theorem then guarantees the existence of a unique u satisfying $u = f(u)$. Obviously it is a solution of (1.2). The uniqueness of this solution follows from the energy inequality (2.17). The proof is completed.

Remark 2.2. Lemma 2.2 shows that $u_{tt} \in C\left([0,\ T];\ H^{-1}(\Omega)\right)$.

3. Global Existence and Energy Decay

In this section, we establish a global existence result and show that the energy of this global solution decays exponentially.

Theorem 3.1. *Suppose that the conditions of Theorem 2.3 hold. Then the solution (2.4) is global; i.e,*

$$u \in C\left([0,\ \infty);\ H_0^1(\Omega)\right) \cap C^1\left([0,\ \infty);\ L^2(\Omega)\right). \tag{3.1}$$

Proof. To establish (3.1), it suffices to show that the solution (2.4) remains bounded, independantly of T, in its space. So we have to prove that there exists a constant K independant of T such that

$$||\nabla u(.,t)||_2^2 + ||u_t(.,t)||_2^2 \le K, \quad \forall\, t \ge 0. \tag{3.2}$$

This is too trivial in our case. We only multiply equation (1.2) by $e^{\Phi(x)}u_t$ and integrate over Ω x $(0,\ t)$ to obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega e^{\Phi(x)}[u_t^2 + |\nabla u|^2](x,t)dx &+ \frac{\beta}{p}\int_\Omega e^{\Phi(x)}|u(x,t)|^p \\ &+ \alpha\int_0^t\int_\Omega e^{\Phi(x)}|u_t(x,s)|^2dxds \\ = \frac{1}{2}\int_\Omega e^{\Phi(x)}[u_1^2 + |\nabla u_0|^2](x,t)dx &+ \frac{\beta}{p}\int_\Omega e^{\Phi(x)}|u_0(x,t)|^p, \quad \forall t \ge 0;\end{aligned}$$

which yields

$$\begin{aligned}&\int_\Omega [u_t^2 + |\nabla u|^2](x,t)dx \\ &\quad \le \frac{A}{a}\left(\int_\Omega [u_1^2 + |\nabla u_0|^2](x,t)dx + \frac{2\beta}{p}\int_\Omega |u_0(x,t)|^p\right) = K, \quad \forall t \ge 0.\end{aligned}$$

Therefore (3.2) is established.

To prove the decay result, we make use of a theorem by Komornik [6], which we state as a lemma without proof.

Lemma 3.2. *Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function such that there exists a constant γ, for which*

$$\int_S^\infty E(t)dt \leq \gamma E(S), \ \forall S \in \mathbb{R}^+. \tag{3.3}$$

Then

$$E(t) \leq E(0)e^{1-t/\gamma}, \ \forall t \geq 0. \tag{3.4}$$

Next we define an 'equivalent' energy of the solution

$$E(t) := \int_\Omega e^{\Phi(x)}[u_t^2 + |\nabla u|^2](x,t)dx + \frac{2\beta}{p}\int_\Omega e^{\Phi(x)}|u(x,t)|^p dx. \tag{3.5}$$

A straightforward calculations show that a multiplication of equation (1.2) by $e^{\Phi(x)}u_t$ and integration over Ω leads to

$$E'(t) = -2\alpha \int_\Omega e^{\Phi(x)}u_t^2(x,t)dx, \tag{3.6}$$

for any regular solution of (1.2). This identity remains valid for solutions (3.1) by a simple density argument. Therefore E is a nonincreasing function.

Theorem 3.3. *Under the conditions Theorem 2.3, the solution (3.1) satisfies*

$$E(t) \leq E(0)e^{1-t/\gamma}, \quad \forall t \geq 0, \tag{3.7}$$

where

$$\gamma = 1 + \frac{2}{\alpha} + \frac{2AC}{a}. \tag{3.8}$$

Here C is a constant depending on Ω only and A and a are the upper and lower bounds of $e^{\Phi(x)}$.

Proof. We multiply equation (1.2) by $e^{\phi(x)}u$ and integrate over Ω to get

$$-\int_\Omega e^{\Phi(x)}u_t^2(x,t)dx + \int_\Omega e^{\Phi(x)}|\nabla u(x,t)|^2 dx + \beta\int_\Omega e^{\Phi(x)}|u(x,t)|^p dx + \frac{d}{dt}\int_\Omega e^{\Phi(x)}uu_t(x,t)dx = -\alpha\int_\Omega e^{\Phi(x)}uu_t(x,t)dx, \tag{3.9}$$

for any regular solution of (1.2). Again this identity remains valid for solutions (3.1) by a simple density argument. By combining (3.5) and (3.9), we arrive at

$$E(t) \le -\frac{\alpha}{2}\frac{d}{dt}\int_\Omega e^{\Phi(x)}u^2(x,t)dx - \frac{d}{dt}\int_\Omega e^{\Phi(x)}uu_t(x,t)dx + 2\int_\Omega e^{\Phi(x)}u_t^2(x,t)dx$$
$$- \beta(1-\frac{2}{p})\int_0^1 e^{\Phi(x)}|u(x,t)|^p dx,$$

which gives, by (3.6) and the condition $p > 2$,

$$E(t) \le -\frac{\alpha}{2}\frac{d}{dt}\int_\Omega e^{\Phi(x)}u^2(x,t)dx - \frac{d}{dt}\int_\Omega e^{\Phi(x)}uu_t(x,t)dx - \frac{1}{\alpha}E'(t). \quad (3.10)$$

We then integrate (3.10) over (S, T) to obtain

$$\int_S^T E(t)dt \le \frac{\alpha}{2}\int_\Omega e^{\Phi(x)}u^2(x,S)dx - \frac{\alpha}{2}\int_\Omega e^{\Phi(x)}u^2(x,T)dx$$
$$+ \int_\Omega e^{\Phi(x)}uu_t(x,S)dx - \int_\Omega e^{\Phi(x)}uu_t(x,T)dx + \frac{2}{\alpha}(E(S)-E(T))$$
$$\le \frac{\alpha}{2}\int_\Omega e^{\Phi(x)}u^2(x,S)dx + \frac{2}{\alpha}E(S) + \int_\Omega e^{\Phi(x)}uu_t(x,S)dx$$
$$- \int_\Omega e^{\Phi(x)}uu_t(x,T)dx, \qquad 0 \le S < T < \infty. \quad (3.11)$$

We now estimate the righthand side of (3.11). By using Poincaré's inequality, we get

$$\int_\Omega e^{\Phi(x)}u^2(x,S)dx \le A\int_\Omega u^2(x,S)dx \le AC\int_\Omega |\nabla u(x,S)|^2dx$$
$$\le \frac{AC}{a}\int_\Omega e^{\Phi(x)}|\nabla u(x,S)|^2dx \le \frac{AC}{a}E(S). \quad (3.12)$$

Also by using Schwarz inequality we have

$$\left|\int_\Omega e^{\Phi(x)}uu_t(x,t)dx\right| \le \frac{1}{2}\int_\Omega e^{\Phi(x)}u^2(x,t)dx + \frac{1}{2}\int_\Omega e^{\Phi(x)}u_t^2(x,t)dx$$
$$\le \frac{1}{2}\{1+\frac{AC}{a}\}E(t) \le \frac{1}{2}\{1+\frac{AC}{a}\}E(S), \quad S \le t \le T. \quad (3.13)$$

Therefore by combining (3.11) - (3.13), we arrive at

$$\int_S^T E(t)dt \le \gamma E(S), \qquad \forall S < T,$$

where γ is defined in (3.8). By letting T go to infinity, (3.3) is verified; hence (3.7) follows and the proof of the theorem is completed.

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