| Nonlinear Differential Equations and Applications NoDEA

Exponential decay of solutions of a nonlinearly damped wave equation

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Abstract. The issue of stablity of solutions to nonlinear wave equations has been addressed by many authors. So many results concerning energy decay have been established. Here in this paper we consider the following nonlinearly damped wave equation

$$u_{tt} - \Delta u + a(1 + |u_t|^{m-2})u_t = bu|u|^{p-2},$$

$a, b > 0$, in a bounded domain and show that, for suitably chosen initial data, the energy of the solution decays exponentially even if $m > 2$.

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1 Introduction

In [9] Messaoudi considered the following problem

$$u_{tt} - \Delta u + g(u_t) + f(u) = 0, \quad x \in \Omega, \quad t > 0$$
$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.1}$$

where $f(u) = bu|u|^{p-2}$, $g(u_t) = a(1 + |u_t|^{m-2})u_t, a, b > 0, m, p > 2$, and Ω is a bounded domain of $\mathbb{R}^n(n \geq 1)$, with a smooth boundary $\partial\Omega$. He showed that, for any initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the problem has a unique global solution with energy decaying exponentially. In the case when $g(u_t) = |u_t|^{m-2}u_t$, Nakao [12] showed that (1.1) has a unique global weak solution if $0 \leq p - 2 \leq 2/(n-2), n \geq 3$ and a global unique strong solution if $p - 2 > 2/(n-2), n \geq 3$ (of course if $n = 1$ or 2 then the only requirement is $p \geq 2$). In addition to global existence the issue of the decay rate was also addressed. In both cases it has been shown that the energy of the solution decays algebraically if $m > 2$ and decays exponentially if $m = 2$. This improves an earlier result by Nakao [11], where he studied the problem in an abstract setting and established a theorem concerning decay of the solution energy only for the case $m - 2 \leq 2/(n-2), n \geq 3$. Also in a joint work, Nakao and Ono [13] extended this result to the Cauchy problem

$$u_{tt} - \Delta u + \lambda^2(x)u + \rho(u_t) + f(u) = 0, \quad x \in \mathbb{R}^n, \quad t > 0$$
$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\rho(u_t)$ behaves like $|u_t|^{\beta}u_t$ and $f(u)$ behaves like $-bu|u|^{\alpha}$. In this case the authors required that the initial data be small enough in the $H^1 \times L^2$ norm and with compact support. Later on, Ono [14] studied the global existence and the decay properties of smooth solutions to the Cauchy problem related to (1.1), for $f \equiv 0$ and gave sharp decay estimates of the solution without any restrictions on the data size. Concerning nonexistence in (1.1), it is well known that if $a = 0$ then the source term $f(u) = -bu|u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [1], [3], [4]). The interaction between the damping and the source terms was first considered by Levine [5] in the linear damping case ($m = 2$). He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [2] extended Levine's result to the nonlinear damping case ($g(u_t) = |u_t|^{m-2}u_t$). In their work the authors introduced a new method and determined suitable relations between m and p, for which there is global existence or alternatively finite time blow up. Precisely they showed that the solutions continue to exist globally 'in time' if $m \geq p$ and blow up in finite time if $m < p$ and the initial energy is sufficiently negative. This result was later generalized to an abstract setting and to unbounded domains by Levine and Serrin [6] and Levine, Pucci, and Serrin [7], and Levine and Park [8]. In these papers, the authors showed that no solution with negative energy can be extended on $[0, \infty)$, if the nonlinearity dominates the damping effect ($p > m$). This generalization allowed them also to apply their result to quasilinear situations. Vitillaro [19] extended these results to situations where the damping is nonlinear and the solution has positive initial energy. It is also worth mentioning that the blow up result of [2] has been improved by Messaoudi [10], where the condition of sufficiently negative has been weakened to negative only. In this paper we are concerned with (1.1),

for $f(u) = -bu|u|^{p-2.}$ and $g(u_t) = a(1 + |u_t|^{m-2})u_t$. Precisely we consider

$$u_{tt} - \Delta u + a(1 + |u_t|^{m-2})u_t = bu|u|^{p-2}, \qquad x \in \Omega, \qquad t > 0$$
$$u(x,t) = 0, \qquad x \in \partial\Omega, t \geq 0$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \qquad (1.3)$$

and show that for suitably chosen initial data, (1.3) possesses a global weak solution, which decays exponentially even if $m > 2$. Our proof of the global existence is based on the use of the potential well theory introduced by Sattiger [16] and Payne and Sattiger [15]. See also Todorova [17], [18] for more recent work. We first state an existence result, which is known as a standard one (see [2]).

Proposition.   Suppose that $m \geq 2, p \geq 2$, such that

$$p \leq 2\frac{n-1}{n-2}, \quad n \geq 3 \qquad (1.4)$$

and let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Then problem (1.3) has a unique solution

$$u \in C([0,T); \ H_0^1(\Omega))$$
$$u_t \in C([0, \ T); \ L^2(\Omega)) \cap L^m(\Omega \ x \ (0, \ T)). \qquad (1.5)$$

for some T small.

Remark 1.1 Condition (1.4) is needed to establish the local existence result (see [2]). In fact, under this condition the nonlinearity is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$. Also from Poincaré's inequality and Sobolev embedding theorems, there exists a constants C_* depending on Ω, m only such that

$$\|u\|_q \leq C_* \|\nabla u\|_2, \quad 2 \leq q \leq \frac{2n}{n-2}, \quad n \geq 3. \qquad (1.6)$$

2   Main result

In order to state and prove our main result we first introduce the following

$$I(t) = I(u(t)) = \|\nabla u(t)\|_2^2 - b\|u(t)\|_p^p$$

$$J(t) = J(u(t)) = \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{b}{p}\|u(t)\|_p^p$$

$$E(t) = E(u(t), u_t(t)) = J(t) + \frac{1}{2}\|u_t(t)\|_2^2$$

$$H = \{w \in H_0^1(\Omega)/I(w) > 0\} \cup \{0\} \qquad (2.1)$$

where we are using $w(t)$ instead of $w(.,t)$.

Remark 2.1 By multiplying equation (1.3) by u_t, integrating over Ω, and using integration by parts, we get

$$E'(t) = -a(\|u_t(t)\|_m^m + \|u_t(t)\|_2^2) \leq 0, \qquad (2.2)$$

for almost each t in $[0, T)$.

Lemma 2.1 *Suppose that*

$$2 < p \leq 2\frac{n-1}{n-2}, \qquad n \geq 3 \qquad (2.3)$$

holds. If $u_0 \in H$ and $u_1 \in L^2(\Omega)$ such that

$$\beta = bC_*^p \left(\frac{2p}{p-2} E(u_0, u_1)\right)^{(p-2)/2} < 1 \qquad (2.4)$$

then $u(t) \in H$, for each $t \in [0, T)$.

Proof. Since $I(u_0) > 0$ then there exists $T_m \leq T$ such that $I(u(t)) \geq 0$ for all $t \in [0, T_m)$. This implies

$$\begin{aligned}
J(t) &= \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{b}{p}\|u(t)\|_p^p \\
&= \frac{p-2}{2p}\|\nabla u(t)\|_2^2 + \frac{1}{p}I(u(t)) \\
&\geq \frac{p-2}{2p}\|\nabla u(t)\|_2^2, \qquad \forall t \in [0, T_m);
\end{aligned} \qquad (2.5)$$

hence

$$\begin{aligned}
\|\nabla u(t)\|_2^2 &\leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \\
&\leq \frac{2p}{p-2}E(u_0, u_1), \qquad \forall t \in [0, T_m).
\end{aligned} \qquad (2.6)$$

By exploiting (1.6), (2.4), and (2.6), we easily arrive at

$$\begin{aligned}
b\|u(t)\|_p^p &\leq bC_*^p\|\nabla u(t)\|_2^p = bC_*^p\|\nabla u(t)\|_2^{p-2}\|\nabla u(t)\|_2^2 \\
&\leq bC_*^p \left(\frac{2p}{p-2}E(u_0, u_1)\right)^{(p-2)/2} \|\nabla u(t)\|_2^2 = \beta\|\nabla u(t)\|_2^2 \\
&< \|\nabla u(t)\|_2^2, \qquad \forall t \in [0, T_m);
\end{aligned} \qquad (2.7)$$

hence $\|\nabla u(t)\|_2^2 - b\|u(t)\|_p^p > 0$, $\forall t \in [0, T_m)$. This shows that $u(t) \in H, \forall t \in [0, T_m)$. By noting that $bC_*^p(\frac{2p}{p-2}E(u(T_m), u_t(T_m)))^{(p-2)/2} < 1$ we easily repeat the steps (2.5)–(2.7) to extend T_m to $2T_m$. We continue this procedure until $u(t) \in H, \forall t \in [0, T)$. $\qquad \square$

Theorem 2.2 *Suppose that* (2.3) *holds. If* $u_0 \in H$ *and* $u_1 \in L^2(\Omega)$ *satisfying* (2.4) *Then the solution is global*

Proof. It suffices to show that $\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2$ is bounded independently of t. To achieve this we use (2.1) and (2.2); so

$$E(u_0, u_1) \geq E(t) = \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{b}{p}\|u(t)\|_p^p + \frac{1}{2}\|u_t(t)\|_2^2$$

$$= \frac{p-2}{2p}\|\nabla u(t)\|_2^2 + \frac{1}{p}I(u(t)) + \frac{1}{2}\|u_t(t)\|_2^2$$

$$\geq \frac{p-2}{2p}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_2^2 \tag{2.8}$$

since $I(u(t)) \geq 0$. Therefore

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq C_1 E(u_0, u_1)$$

for $C = \max\{2, 2p/(p-2)\}$. $\qquad\square$

Remark 2.2 If $m \geq p$ then the global existence can be obtained for any $u_0 \in H_0^1(\Omega)$ and any $u_1 \in L^2(\Omega)$. See [2].

Lemma 2.3 *Suppose that*

$$2 < m \leq \frac{2n}{n-2}, \qquad n \geq 3. \tag{2.9}$$

Then the solution satisfies

$$\|u(t)\|_m^m \leq CE(t), \tag{2.10}$$

for some constant C independent of t.

Proof.

$$\|u(t)\|_m^m \leq C_*^m \|\nabla u(t)\|_2^m \leq C_*^m \|\nabla u(t)\|_2^{m-2}\|\nabla u(t)\|_2^2$$

$$\leq C_*^m \left(\frac{2p}{p-2}E(u_0, u_1)\right)^{(m-2)/2} \frac{2p}{p-2}E(t)$$

by virtue of (1.5) and (2.8). Therefore (2.10) is established. $\qquad\square$

Theorem 2.4 *Suppose that* (2.3), (2.4), *and* (2.9) *hold. Then there exist positive constants K and k such that the global solution of* (1.3) *satisfies*

$$E(t) \leq Ke^{-kt}, \quad \forall t \geq 0. \tag{2.11}$$

Proof. We define

$$F(t) := E(t) + \varepsilon \int_\Omega \left(u(t)u_t(t) + \frac{a}{2}u^2(t) \right) dx, \tag{2.12}$$

for ε so small that

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t), \tag{2.13}$$

holds for two positive constants α_1 and α_2. We differentiate (2.12) and use equation (1.3) to obtain

$$\begin{aligned}
F'(t) = &-a \int_\Omega |u_t(t)|^2 dx - a \int_\Omega |u_t(t)|^m dx + \varepsilon \int_\Omega [u_t^2(t) - |\nabla u(t)|^2] dx \\
&- a\varepsilon \int_\Omega |u_t(t)|^{m-2} u_t(t) u(t) dx + \varepsilon b \int_\Omega |u(t)|^p dx \\
\le &-a \int_\Omega |u_t(t)|^m dx - [a - \varepsilon] \int_\Omega u_t^2(t) dx - \varepsilon \int_\Omega |\nabla u(t)|^2 dx \\
&+ a\varepsilon \int_\Omega |u_t(t)|^{m-1} |u(t)| dx + \varepsilon b \int_\Omega |u(t)|^p dx.
\end{aligned} \tag{2.14}$$

We then use (2.1) and (2.7) to get

$$\begin{aligned}
b \int_\Omega |u(t)|^p dx = &\alpha b \int_\Omega |u(t)|^p dx + (1-\alpha) b \int_\Omega |u(t)|^p dx \\
= &\alpha \left(\frac{p}{2} \int_\Omega u_t^2(t) dx + \frac{p}{2} \int_\Omega |\nabla u(t)|^2 dx - pE(t) \right) \\
&+ (1-\alpha)\beta \int_\Omega |\nabla u(t)|^2 dx, \quad 0 < \alpha < 1
\end{aligned} \tag{2.15}$$

and exploit Young's inequality to estimate

$$\int_\Omega |u_t(t)|^{m-1} |u(t)dx| \le \delta \|u(t)\|_m^m + c(\delta)\|u(t)_t\|_m^m, \qquad \forall \delta > 0. \tag{2.16}$$

Therefore a combination of (2.14)–(2.16) gives

$$\begin{aligned}
F'(t) \le &-a \int_\Omega |u_t(t)|^m dx - \left[a - \varepsilon \left(\frac{\alpha p}{2} + 1 \right) \right] \int_\Omega u_t^2(t) dx - \alpha p E(t) \\
&+ \varepsilon \left[\alpha \left(\frac{p}{2} - 1 \right) - \eta(1-\alpha) \right] \int_\Omega |\nabla u(t)|^2 dx \\
&+ \varepsilon a \left(\delta \|u(t)\|_m^m + c(\delta)\|u_t(t)\|_m^m \right)
\end{aligned} \tag{2.17}$$

where $\eta = 1 - \beta$. By using (2.8) and choosing α close to 1 so that $\alpha(\frac{p}{2} - 1) - \eta(1 - \alpha) \geq 0$, we arrive at

$$
\begin{aligned}
F'(t) \leq{} & -a \int_\Omega |u_t(t)|^m dx - \left[a - \varepsilon \left(\frac{\alpha p}{2} + 1\right)\right] \int_\Omega u_t^2(t) dx - \alpha p E(t) \\
& + \varepsilon \left[\alpha \left(\frac{p}{2} - 1\right) - \eta(1 - \alpha)\right] \frac{2p}{p-2} E(t) \\
& + \varepsilon a \left(\delta \|u(t)\|_m^m + c(\delta) \|u_t(t)\|_m^m\right) \\
\leq{} & -a \int_\Omega |u_t(t)|^m dx - \left[a - \varepsilon \left(\frac{\alpha p}{2} + 1\right)\right] \int_\Omega u_t^2(t) dx \\
& - \eta \varepsilon(1 - \alpha) \frac{2p}{p-2} E(t) + \varepsilon a \left(\delta \|u(t)\|_m^m + c(\delta) \|u_t(t)\|_m^m\right) \qquad (2.18)
\end{aligned}
$$

We then recall Lemma 2.3 to substitute for $\|u(t)\|_m^m$; hence (2.18) becomes

$$
\begin{aligned}
F'(t) \leq{} & -a \left[1 - \varepsilon c(\delta)\right] \|u_t(t)\|_m^m - \left[a - \varepsilon \left(\frac{\alpha p}{2} + 1\right)\right] \int_\Omega u_t^2(t) dx \\
& - \varepsilon \left[\eta(1 - \alpha) \frac{2p}{p-2} - \delta a C\right] E(t). \qquad (2.19)
\end{aligned}
$$

At this point we choose δ so small that $\eta(1 - \alpha)\frac{2p}{p-2} - \delta a C > 0$. Once δ is chosen we then pick ε so small that $1 - \varepsilon c(\delta) \geq 0, a - \varepsilon(\frac{\alpha p}{2} + 1) \geq 0$, and (2.13) remains valid. Consequently (2.19) yields

$$
\begin{aligned}
F'(t) &\leq -\varepsilon \left[\eta(1 - \alpha) \frac{2p}{p-2} - \delta a C\right] E(t) \\
&\leq -\varepsilon \alpha_2 \left[\eta(1 - \alpha) \frac{2p}{p-2} - \delta a C\right] F(t) \qquad (2.20)
\end{aligned}
$$

by virtue of (2.13). A simple integration of (2.20) then leads to

$$
F(t) \leq F(0)e^{-kt}, \qquad (2.21)
$$

where $k = \varepsilon \alpha_2[\eta(1 - \alpha)\frac{2p}{p-2} - \delta a C]$ Again using (2.13) we obtain (2.11). This completes the proof. $\qquad \square$

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