Local existence and blow up in a semilinear heat equation with the Bessel operator

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Abstract In this work we consider an initial one-point boundary value problem to the heat equation with the Bessel operator $u_t - (u_{xx} + \frac{1}{x}u_x) = |u|^{p-2}u$.
We first prove a local existence result. Then we show that the solution blows up in finite time.

1 Introduction

In [3] Denche and Marhoune considered the following linear problem

$$u_t - (u_{xx} + \tfrac{1}{x}u_x) = f(x,t), \qquad x \in (0,l), \ \ 0 < t < T,$$
$$u(l,t) = 0, \quad \int_{l_1}^{l} u(x,t)dx = 0, \quad 0 \le l_1 \le l, 0 < t < T$$
$$u(x,0) = \phi(x), \qquad x \in I.$$

and proved the existence of a strong solution. Their work relatively improves earlier results by Benuar and Yurchuk [2]. Also a similar problem containing $\partial_x(a(x,t)u_x)$ instead of the Bessel operator, has been investigated by Kartynnik [4] and Yurchuk [5]. In these papers the methods of proof were essentially based on operator techniques. The authors defined an operator L from a space E into another space F, and then showed that L is a linear homeomophism. This allowed them to prove their existence results.

In this paper we are concerned with the local existence and the finite time blow up of weak solutions of a semilinear one-point boundary value problem for the heat equation with the Bessel operator. So we consider the following problem,

$$u_t - (u_{xx} + \tfrac{1}{x}u_x) = |u|^{p-2}u, \qquad x \in I = (0,1), \quad t > 0,$$
$$u(1,t) = 0, \qquad t \ge 0 \qquad\qquad (1.1)$$
$$u(x,0) = \phi(x), \qquad x \in I,$$

where $\phi \in H$. Here, H is the Banach space obtained by completing the space $F = \{v \in C^1([0,1]) \ / \ v(1) = 0\}$ with respect to the norm

$$||v||_H^2 = \int_0^1 x[v^2(x) + |v'(x)|^2]dx. \tag{1.2}$$

We also define the weighted Banach space E to contain all functions v satisfying

$$||v||_E^2 = \int_0^1 xv^2(x)dx < \infty. \tag{1.3}$$

Problem (1.1) is obtained from the study of the radial solutions of the following two-dimensional heat problem

$$\begin{aligned}
u_t - (u_{yy} + u_{zz}) &= |u|^{p-2}u, & (y,z) \in D, \quad t > 0, \\
u &= 0, & (y,z) \in \partial D, \quad t \geq 0 \\
u(y,z,0) &= \phi(y,z), & (y,z) \in D,
\end{aligned}$$

where D is the unit disk centered at the origin.

In order to establish our local existence result for (1.1), we first prove an existence theorem of weak solutions for the related linear problem.

2   The linear Problem

In this section we consider the linear problem

$$\begin{aligned}
u_t - (u_{xx} + \tfrac{1}{x}u_x) &= f(x,t), & x \in I, \quad t \in (0,T), \\
u(1,t) &= 0, & t \in [0,T] \\
u(x,0) &= \phi(x), & x \in I,
\end{aligned} \tag{2.1}$$

where $f \in L^2(0,T;E)$, ie $\int_0^T \int_0^1 x(f(x,t))^2 dxdt < \infty$. We start with a lemma that gives an equivalent norm to (1.2).

Lemma 2.1. *For v in H, we have*

$$\int_0^1 xv^2(x)dx \leq 4\int_0^1 x(v_x(x))^2 dx. \tag{2.2}$$

Proof. It is easy to see that for each v in F we have

$$0 = \int_0^1 (xv^2)_x dx = \int_0^1 (v^2 + 2xvv_x)dx;$$

hence we get

$$\int_0^1 xv^2 dx \leq \int_0^1 v^2 dx = -2\int_0^1 xvv_x dx.$$

By using Young's inequality we obtain

$$\int_0^1 xv^2 dx \leq |-2\int_0^1 xvv_x dx| \leq 2\int_0^1 xv_x^2 dx + \frac{1}{2}\int_0^1 xv^2 dx.$$

Therefore (2.2) is established for each v in F. This inequality remains valid for v in H since F is dense in H.

Remark This is Poincare's inequality for the space H.

Theorem 2.2. *Let f be in $L^2(0,T;E)$. Then problem (2.1) has a unique weak solution*

$$u \in L^\infty(0,T;H), \qquad u_t \in L^2(0,T;E). \tag{2.3}$$

Moreover $\forall\, t \in [0,\ T]$, we have

$$\int_0^1 x{u_x}^2(x,t)dx + \int_0^t \int_0^1 xu_t^2(x,s)dxds \leq$$

$$C\int_0^1 x(\phi_x(x))^2 dx + C \int_0^t \int_0^1 x|f(x,s)||u_t(x,s)|dxds, \tag{2.4}$$

where C is a constant independent of u and t.

Proof. By multiplying equation (2.1) by x we obtain

$$(xu)_t - (xu_x)_x = xf(x,t). \tag{2.5}$$

We then use Galerkin method to prove our theorem. For this purpose let $(e_i)_{i=1}^\infty$ be a basis of H {H is a Hilbert space}and H_m be the finite dimensional subspace spanned by $\{e_1, e_2, ..., e_m\}$. We would like to approximate the solution by functions lying in these spaces, having the forms

$$u_m(x,t) := \sum_{i=1}^m \alpha_i(t)e_i \tag{2.6}$$

and satisfying

$$\sum_{i=1}^m \alpha_i'(t) \int_0^1 xe_ie_j dx + \sum_{i=1}^m \alpha_i(t) \int_0^1 xe_i'e_j' dx =$$

$$\int_0^1 xf(x,t)e_j dx, \quad 0 < t < T, \quad j = 1,2,...m \tag{2.7}$$

$$\sum_{i=1}^m \alpha_i(0) \int_0^1 xe_ie_j dx = \int_0^1 x\phi(x)e_j dx, \quad j = 1,2,...m. \tag{2.8}$$

The standard theory for the Ordinary Differential Equations guarantees the existence of functions $\alpha_i's$ such that u_m, defined by (2.6), satisfies (2.7), (2.8). We then substitute e_j by $\partial u_m/\partial t$ in (2.7) and integrate over $(0,t)$ to get

$$\frac{1}{2}\int_0^1 x(\partial_x u_m(x,t))^2 dx + \int_0^t \int_0^1 x(\partial_t u_m(x,s))^2 dxds \leq \frac{1}{2}\int_0^1 x(\phi_x(x))^2 dx$$

$$+ \int_0^t \int_0^1 x|f(x,s)||\partial_t u_m(x,s)|dxds, \quad \forall\, t \in [0,\ T]. \tag{2.9}$$

By using the Schwarz inequality we conclude that (u_m) is bounded in $L^\infty(0,T;H)$ and $(\partial_t u_m)$ is bounded in $L^2(0,T;E)$; so we can extract subsequences { still denoted by (u_m) and $(\partial_t u_m)$}such that u_m converges weakly $*$ to a function u in $L^\infty(0,T;H)$ and $\partial_t u_m$ converges weakly to u_t in $L^2(0,T;E)$.

Since, by virtue of (2.7), we have

$$\frac{d}{dt}\int_0^1 xu_m(x,t)e_j dx + \int_0^1 x\partial_x u_m(x,t)e_j' dx = \int_0^1 xf(x,t)e_j dx,$$

for each $m \geq j$ then it is easy to see that, for each $j \geq 1$ and for almost every $t \in [0,T]$, u satisfies

$$\frac{d}{dt}\int_0^1 xu(x,t)e_j dx + \int_0^1 xu_x(x,t)e_j' dx = \int_0^1 xf(x,t)e_j dx. \qquad (2.10)$$

Therefore, for each $\psi \in H$ and for almost every $t \in [0,T]$, we get

$$\frac{d}{dt}\int_0^1 xu(x,t)\psi(x)dx + \int_0^1 xu_x(x,t)\psi'(x)dx = \int_0^1 xf(x,t)\psi(x)dx. \qquad (2.11)$$

The uniqueness can be established in the usual way by supposing the existence of two solutions having for difference w which satisfies

$$(xw)_t - (xw_x)_x = 0, \qquad w(x,0) = 0. \qquad (2.12)$$

By multiplying (2.11) by w and integrating over I we get

$$\frac{d}{dt}\int_0^1 xw^2(x,t)dx \leq -\int_0^1 xw_x^2(x,t)dx,$$

which yields

$$\int_0^1 xw^2(x,t)dx \leq \int_0^1 xw^2(x,0)dx = 0$$

Therefore $\int_0^1 xw^2(x,t)dx = 0$; hence $w \equiv 0$.
The estimate (2.4) is a direct result of (2.9) and the sequential lower semicontinuity of the norm function.

3   The Semilinear Problem

In this section we state and prove the local existence result to problem (1.1). We first start with the following

Lemma 3.1 *If $v \in H$ and $2 < p < 3$ then $|v|^{p-2}v \in E$.*

Proof First we note that by virtue of lemma 5.42 of [1] and using a density argument we have

$$\sup\{x(v(x))^2, 0 < x < 1\} \leq 4\int_0^1 xv^2(x)dx + 4\int_0^1 x|v(x)||v'(x)|dx.$$

By using the Schwarz inequality and lemma 2.1, this yields

$$\sup\{x|v(x)|^2, 0 < x < 1\} \leq C \int_0^1 x|v'(x)|^2 dx. \tag{3.1}$$

By evaluating the E-norm of $|v|^{p-2}v$ we have

$$\int_0^1 x|v(x)|^{2p-2}dx = \int_0^1 x^{p-1}|v(x)|^{2(p-1)}x^{2-p}dx$$

$$\leq \left(\sup\{x|v(x)|^2, 0 < x < 1\}\right)^{p-1}\int_0^1 x^{2-p}dx$$

$$\leq \frac{C}{3-p}(\|v\|_H)^{2p-2} < \infty,$$

by virtue of (3.1). This completes the proof.

Theorem 3.2 *Let $\phi \in H$ and $2 < p < 3$, then for $T^* < T$ problem (1.1) has a unique local solution u with the property*

$$u \in L^\infty(0, T^*; H), \qquad u_t \in L^2(0, T^*; E) \tag{3.2}$$

Proof We prove this theorem by using a fixed point argument. For $T > 0$ and $M > 0$, we define the class of functions $W = W(M, T) = \{w \in L^\infty(0, T; H)$ with $w(x, 0) = \phi\}$ such that

$$N(w, w_t, T) = \sup\{\|w(., t)\|_H^2, 0 < t < T\} + \int_0^T \int_0^1 x|w_t(x, t)|^2 dxdt \leq M^2. \tag{3.3}$$

We then define a map $h : W \rightarrow H$ which associates to each $v \in W$ the solution u of the linear problem

$$\begin{aligned} u_t - (u_{xx} + \tfrac{1}{x}u_x) &= |v|^{p-2}v, & x \in I, \quad t > 0 \\ u(1, t) &= 0, & t \geq 0 \\ u(x, 0) &= \phi(x), & x \in I. \end{aligned} \tag{3.4}$$

It follows from lemma 3.1 and theorem 2.2 that (3.4) has a unique solution u satisfying

$$\int_0^1 x{u_x}^2(x, t)dx + \int_0^t \int_0^1 xu_t^2(x, s)dxds \leq$$

$$C\int_0^1 x|\phi'(x)|^2dx + C\int_0^t \int_0^1 x|v(x, s)|^{2p-2}dxds, \quad \forall\, t \in [0, T].$$

This, in turn, implies that

$$\sup\{\|u(., t)\|_H^2, 0 < t < T\} + \int_0^T \int_0^1 xu_t^2(x, t)dxdt \leq \Gamma + CTM^{2p-2}, \tag{3.5}$$

where Γ is a constant depending on the H-norm of ϕ only. By taking M large enough and T^* small enough, (3.5) yields

$$N(u, u_t, T^*) \le M^2;$$

hence h maps W into itself. To show that h is a contraction for T^* small enough, we consider $v_1, v_2 \in W$ and the corresponding images u_1 and u_2. It is straightforward to see that $U = u_1 - u_2$ satisfies

$$\begin{aligned}
U_t - (U_{xx} + \tfrac{1}{x}U_x) &= |v_1|^{p-2}v_1 - |v_2|^{p-2}v_2, \qquad x \in I, \quad t > 0 \\
U(1,t) &= 0, \qquad t \ge 0 \\
U(x,0) &= 0, \qquad x \in I.
\end{aligned} \tag{3.6}$$

We multiply (3.6) by xU_t and integrate over $I \times (0,t)$ to get

$$\frac{1}{2}\int_0^1 xU_x{}^2(x,t)dx + \int_0^t \int_0^1 xU_t^2(x,s)dxds \le$$

$$\int_0^t \int_0^1 x|U_t|||v_1|^{p-2}v_1 - |v_2|^{p-2}v_2|(x,s)dxds.$$

Schwarz inequality then leads to

$$\int_0^1 xU_x{}^2(x,t)dx + \int_0^t \int_0^1 xU_t^2(x,s)dxds \le$$

$$\int_0^t \int_0^1 x\{|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\}^2(x,s)dxds. \tag{3.7}$$

We now estimate the RHS of (3.7) as follows. Taking $V = v_1 - v_2$, we obtain

$$\int_0^1 x\{|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\}^2 dx \le C_1 \int_0^1 x|V|^2\{|v_1|^{2p-4} + |v_2|^{2p-4}\},$$

where C_1 is a constant independent of v_1, v_2 and t. Thus we have, by virtue of (3.1),

$$\int_0^1 x\{|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\}^2 dx$$

$$\le C_1 \left(\sup\{x|V|^2, 0 < x < 1\}\right)\int_0^1 \{|v_1|^{2p-4} + |v_2|^{2p-4}\}dx$$

$$\le C \left(\int_0^1 x|V_x|^2 dx\right)\int_0^1 \{|v_1|^{2p-4} + |v_2|^{2p-4}\}dx, \tag{3.8}$$

Next we evaluate

$$\int_0^1 |v_1|^{2p-4} = \int_0^1 x^{p-2}|v_1|^{2p-4}x^{2-p}dx$$

$$\le \left(\sup\{x|v_1|^2, 0 < x < 1\}\right)^{p-2}\int_0^1 x^{2-p}dx$$

$$\leq C \frac{1}{3-p} \left[\int_0^1 x(\frac{\partial v_1}{\partial x})^2 dx \right]^{p-2} \leq CM^{2(p-2)}. \qquad (3.9)$$

By combining (3.8) and (3.9) we arrive at

$$\int_0^{T^*} \int_0^1 x\{|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\}^2 dx ds \quad \leq \qquad (3.10)$$
$$CT^* M^{2(p-2)} \sup\{||V(.,t)||_H^2, 0 \quad < \quad t < T\}$$

Therefore (3.7) and (3.10) give

$$N(U, U_t, T^*) \leq CT^* M^{2(p-2)} N(V, V_t, T^*). \qquad (3.11)$$

Choosing T^* small enough that $CT^* M^{2(p-2)} < 1$, makes the map h a contraction from W into itself. The Contraction Mapping Theorem then guarantees the existence of a fixed point u, which is the desired solution of (1.1). The proof is then complete.

4 The Finite Time Blow Up

In this section we show that the solution (3.2) blows up in finite time if

$$E_0 := \frac{1}{2} \int_0^1 x(\phi_x(x))^2 dx - \frac{1}{p} \int_0^1 x|\phi(x)|^p dx \leq 0. \qquad (4.1)$$

Theorem 4.1. *Let $2 < p < 3$ then for any nonzero $\phi \in H$ satisfying (4.1), the solution, given in (3.2), blows up in finite time.*
Proof.
    If we set

$$G(t) := -\frac{1}{2} \int_0^1 x(u_x(x,t))^2 dx + \frac{1}{p} \int_0^1 x|u(x,t)|^p dx,$$

multiply equation (1.1) by $-xu_t$ and integrate over I we get

$$G'(t) = \int_0^1 x u_t^2(x,t) dx \geq 0,$$

for any regular solution of (1.1). This identity remains valid for the solution (3.2) by a simple density argument. So we have

$$0 \leq -E_0 = G(0) \leq G(t). \qquad (4.2)$$

We then define

$$L(t) := \frac{1}{2} \int_0^1 xu^2(x,t) dx \qquad (4.3)$$

and differentiate to obtain

$$L'(t) = \int_0^1 x u u_t(x,t) dx = \int_0^1 xu(u_{xx} + \frac{1}{x}u_x + |u|^{p-2}u)(x,t) dx.$$

$$= \int_0^1 x|u|^p dx - \int_0^1 x(u_x)^2 dx = 2G(t) + (1 - \frac{2}{p}) \int_0^1 x|u|^p dx$$

$$\geq (1 - \frac{2}{p}) \left[G(t) + \int_0^1 x|u|^p dx \right]. \tag{4.4}$$

Next we estimate

$$L^{p/2}(t) \leq 2^{-p/2} \left[\int_0^1 xu^2 dx \right]^{p/2} \leq C \left[(\int_0^1 x|u|^p dx)^{2/p} (\int_0^1 xdx)^{(p-2)/p} \right]^{p/2}$$

$$L^{p/2}(t) \leq 2 \int_0^1 x|u|^p dx \leq 2 \left[G(t) + \int_0^1 x|u|^p dx \right]. \tag{4.5}$$

A combination of (4.4) and (4.5) then yields

$$L^{p/2}(t) \leq \frac{2p}{p-2} L'(t). \tag{4.6}$$

We then integrate (4.6) over $(0, t)$ to get

$$L^{p/2-1}(t) \geq \frac{1}{L^{1-p/2}(0) - \gamma t}, \tag{4.7}$$

where $\gamma = (p^2 - 4)/4p$.. Therefore (4.7) shows that L blows up in a time $T^* \leq L^{1-p/2}(0)/\gamma$. This completes the proof.

From the previous result we have the following

Corollary 4.2. *If there exists* $t_0 \geq 0$, *for which*

$$\frac{1}{2} \int_0^1 x|u_x(x, t_0)|^2 dx - \frac{1}{p} \int_0^1 x|u(x, t_0)|^p dx = 0 \tag{4.8}$$

then the solution, given by (3.2), either remains equal to zero for all time $t \geq t_0$ *or blows up in finite time* $t^* > t_0$.

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